Zero-Knowledge Proofs of Knowledge

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Knowledge – Motivation

- Prove that you know the shortest path from A to B
  - A shortest path exists, but who says that you know it?
- Prove identity:
  - For public key $h = g^x$ in a group where discrete log is hard, prove that I know x
  - This proves identity since it is my private key and only I know it
  - Attempt: prove in ZK that $h \in L$ for $L = \{h \mid \exists x: g^x = h\}$
  - Problem:
    - This statement is TRUE for all group elements (and so ZK is actually trivial – send YES)
    - Who says that I need to know a witness to prove a true statement

What is Knowledge?

- **Definition**: a student knows the material if she can output it
  - We approximate this by saying that a student knows the material if she can output the answers to the questions on the test

- **Definition**: a machine knows something if it can output it
  - Let R be an NP-relation
  - A machine knows the witness to a statement x if it can output w s.t. $(x, w) \in R$

- What does it mean for a machine to be able to output it?

Formalizing Knowledge (first attempt)

- Attempt 1: a machine M knows the witness to a statement x if there exists some M' who outputs w s.t. $(x, w) \in R$
- Questions:
  - How does this relate to the machine's actions (e.g., proving a proof)?
  - How is M' related to M; if there is no connection then why does M know it?

Formalizing Knowledge (second attempt)

- Attempt 2:
  - We define a PPT oracle machine K, called a knowledge extractor
  - We say that M knows the witness to a statement x if $K^{M(\cdot)}(x)$ outputs w s.t. $(x, w) \in R$
    - K interacts with M and can use whatever it does to obtain w
    - Since K is generic, its ability to output w means that M knows w
- Questions:
  - This still doesn't relate to the machine's actions (e.g., proving a proof)?
  - K could still just know w independently of M

Formalizing Knowledge (third attempt)

- **Definition**:
  - We define a PPT oracle machine K, called a knowledge extractor
  - We say that a prover P^* knows the witness to a statement x if $K^{P^*(\cdot)}(x)$ outputs w s.t. $(x, w) \in R$ whenever P^* convinces V of x
- **Intuition**:
  - K is generic and works for any x and any P^*: if P^* can convince V then K can output w and so M knows w
- **Question**: what does it mean: “whenever P^* convinces V of x”?
  - K should run in (expected) polynomial-time and output a witness w with the same probability that P^* convinces V of x

Formalizing Knowledge (final)

- One can always prove in ZK without knowing, with negligible prob
  - Run the zero-knowledge simulator and hope that the verifier's queries in the result match the real queries
- The definition is updated to allow a **knowledge error** κ, which takes this into account
  - If P^* convinces V of x with probability $> \kappa$, then K should run in (expected) polynomial-time and output a witness w with probability at most κ less than P^* convinces V of x
- This property is called **knowledge soundness**

The Definition

- **Definition (knowledge soundness)**:
  - A proof system has **knowledge soundness** with error κ if there exists a PPT K s.t. for every prover P^*, if P^* convinces V of x with probability $\epsilon > \kappa$, then $K^{P^*(\cdot)}(x)$ outputs w s.t. $(x, w) \in R$ with probability at least $\epsilon(|x|) - \kappa(|x|)$

An Alternative Formulation

- **Motivation:** one can trade off running time and success probability
  - Definition says: run in PPT and output w.p. ϵ
  - Alternative definition: run in **expected** time $\frac{1}{\epsilon}$ and always output

- **Definition (knowledge soundness)**:
  - A proof system has **knowledge soundness** with error κ if there exists a K s.t. for every prover P^*, if P^* convinces V of x with probability $\epsilon > \kappa$, then $K^{P^*(\cdot)}(x)$ outputs w s.t. $(x, w) \in R$ in expected time $\frac{poly(|x|)}{\epsilon(|x|)-\kappa(|x)}$

Equivalence of the Definitions

- **Original implies alternative:**
  - We are given K that runs in PPT and outputs a witness w.p. ϵ
  - We can run K many times until a witness is output
    - Since it is an **NP relation**, can verify when get correct result
    - Expected number of times needed is $1/\epsilon$

- **Alternative implies original:**
  - We are given K that runs in time $1/\epsilon$ and outputs a witness
  - For $i = 1, \ldots, n$, run K for 2^{i+1} steps; if finish output witness; else proceed w.p. $\frac{1}{2}$
    - Let i be smallest s.t. $2^{i+1} > 1/\epsilon$: probability of getting here is at least $2^{-(i+1)} > \epsilon$
    - Expected running time is $poly(|x|)$

Definition of ZKPOK

- **A proof system is a zero-knowledge proof of knowledge if it has**
  - **Completeness**: honest prover convinces honest verifier
  - **Zero knowledge**: ensures verifier learns nothing
  - **Knowledge soundness**: ensures prover knows witness
- **Zero knowledge is a property of the prover**
  - Prover behavior is guaranteed to reveal nothing
  - Protect against a cheating verifier
- **Knowledge soundness is a property of the verifier**
  - Verifier behavior guarantees that prover knows witness
  - Protect against a cheating prover

Reducing Knowledge Error

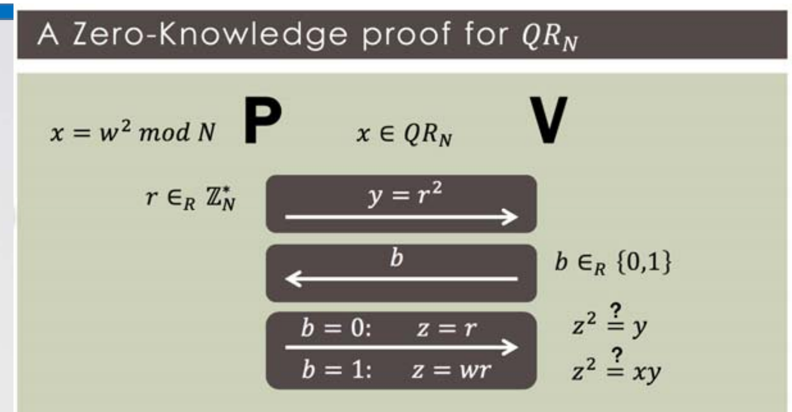
- **Sequential composition reduces knowledge error exponentially**
- **Exponentially small error = zero error**
  - Assume knowledge error $\kappa < 2^{-|x|}$ and consider alternative definition
  - Run $K^{P^*(\cdot)}(x)$ in parallel to running a brute-force search on witness
    - Assume brute force in time $2^{|x|}$
  - Let P^* be s.t. it convinces V of x with probability ϵ
    - If $\epsilon > 2 \cdot \kappa$ then $\frac{poly(|x|)}{\epsilon-\kappa} < \frac{2 \cdot poly(|x|)}{\epsilon}$ and so succeed in time $\frac{poly'(|x|)}{\epsilon}$
    - If $\epsilon < 2 \cdot \kappa$ then $\frac{poly(|x|)}{\epsilon} > 2^{|x|} \cdot poly(|x|)$ and so brute force finishes

Constructing ZKPOKs

A Zero-Knowledge proof for QR_N

$x = w^2 \bmod N$ **P**	$x \in QR_N$	**V**
$r \in_R \mathbb{Z}_N^*$	$y = r^2$ $\longrightarrow$	
	b $\longleftarrow$	$b \in_R \{0,1\}$
	$b = 0$: $z = r$ $\longrightarrow$ $b = 1$: $z = wr$	$z^2 \stackrel{?}{=} y$ $z^2 \stackrel{?}{=} xy$

Knowledge Extraction Idea



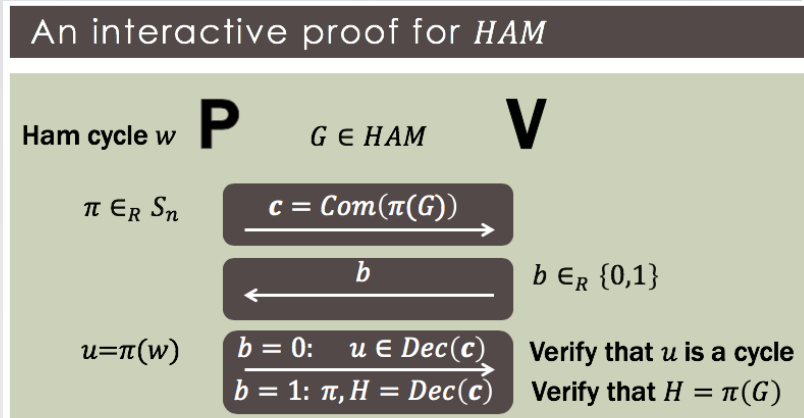
- K invokes P^* and "receives" some y
- K "sends" P^* the query $b = 0$ and receives z_0
- K **rewinds** and "sends" P^* the query $b = 1$ and receives z_1
- K outputs $w = \frac{z_1}{z_0} \bmod N$
- **Proof:**
  - If P^* convinces w.p. greater than $\kappa = \frac{1}{2}$ then $(z_0)^2 = y$ and $(z_1)^2 = xy$
    - I am assuming for deterministic P^*; to discuss!
  - Thus $w^2 = \left(\frac{z_1}{z_0}\right)^2 = \frac{xy}{y} = x$ and so K outputs a square root

ZKPOK for NP

An interactive proof for HAM

Ham cycle w **P**	$G \in HAM$	**V**
$\pi \in_R S_n$	$c = Com(\pi(G)) \longrightarrow$	
	$\longleftarrow b$	$b \in_R \{0,1\}$
$u = \pi(w)$	$b = 0$: $u \in Dec(c) \longrightarrow$	Verify that u is a cycle
	$b = 1$: $\pi, H = Dec(c)$	Verify that $H = \pi(G)$

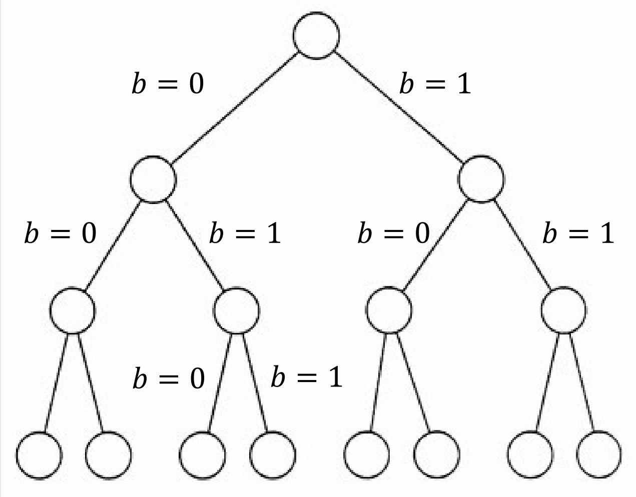
ZKPOK for NP



- K invokes P^* and receives a commitment c
- K sends P^* the query $b = 0$ and receives a cycle w
- K rewinds and sends P^* the query $b = 1$ and receives $\pi, \tilde{G}$
- **Proof:**
  - If P^* convinces w.p. greater than $\kappa = \frac{1}{2}$ then w is a cycle in $\tilde{G} = \pi(G)$
  - Thus, $\pi^{-1}(w)$ is a Hamiltonian cycle in G

ZKPOK for NP with Negligible Error

- Run Hamiltonicity $n = |x|$ times sequentially
- Extractor strategy:
  - Consider binary tree of execution
  - Attempt to extract in ith execution
    - If P^* answers both queries, get Hamiltonian cycle
    - If P^* answers neither query, V always rejects
    - If P^* answers exactly one query, go down that edge
  - Repeat with next execution



- Extraction fails iff P^* answers **exactly one** query in each execution
- Thus, extraction works with probability 1 if $\epsilon > 2^{-n}$

Strong Proofs of Knowledge

- **Definition – strong knowledge soundness**
  - A proof system has **strong knowledge soundness** if there exists a negligible function μ and a PPT K s.t. for every prover P^*, if P^* convinces V of x with probability $\epsilon > \mu$, then $K^{P^*(\cdot)}(x)$ outputs w s.t. $(x, w) \in R$ with probability at least $1 - \mu(|x|)$

- **Theorem**: sequential Hamiltonicity is a strong proof of knowledge

Using the Alternative Definition

- **Definition (knowledge soundness)**:
  - A proof system has **knowledge soundness** with error κ if there exists a K s.t. for every prover P^*, if P^* convinces V of x with probability $\epsilon > \kappa$, then $K^{P^*(\cdot)}(x)$ outputs w s.t. $(x, w) \in R$ in expected time $\frac{poly(|x|)}{\epsilon(|x|)-\kappa(|x)}$

- What does it help to run in time $\frac{poly(|x|)}{\epsilon(|x|)}$ when this may not be polynomial time?

Using the Alternative Definition

- **A classic use of zero-knowledge proofs of knowledge**:
  - Within a protocol, prover proves the proof
  - To prove security, a simulator (or reduction) needs the witness
    - Unless verifier would reject, in which case it doesn't matter
- **Using ZKPOKs in proofs of security – simulator (or reduction) plays verifier with prover**:
  - If the verifier rejects, then the simulator can halt, since a real verifier would
  - If the verifier accepts, then the simulator now has to extract the witness

ZKPOK Inside a Protocol

- **Recall simulator (reduction) strategy**:
  - Verify, then halt if reject and extract if accept

- What is the expected running time of this simulator (reduction)?
  - Probability that prover convinces verifier is $\epsilon(|x|)$
  - Assuming that the knowledge error κ is 0:

$$E[\text{Time}] = \big(1 - \epsilon(|x|)\big) \cdot poly(|x|) + \epsilon(|x|) \cdot \frac{poly(|x|)}{\epsilon(|x|)} = poly(|x|)$$

  - Assuming that the knowledge error κ is negligible:

$$E[\text{Time}] = \big(1 - \epsilon(|x|)\big) \cdot poly(|x|) + \epsilon(|x|) \cdot \frac{poly(|x|)}{\epsilon(|x|)-\kappa(|x)} = poly(|x|) + \frac{\epsilon(|x|)}{\epsilon(|x|-\kappa(|x|)}$$

    - Actually not polynomial, but can be fixed…

ZKPOK in a Protocol

- **The issue that arises is that need to both**
  - Simulate the view of the prover in the execution, **and**
  - Extract a witness
- This is called "**witness-extended emulation**"
- A witness-extended emulator $E^{P^*(\cdot)}(x)$ outputs a VIEW and some w:
  - The view output is indistinguishable from a real execution
  - The probability that the view is accepting and yet $(x, w) \notin R$ is negligible
  - E runs in expected polynomial-time

Witness-Extended Emulation

- **Lemma**: If (P, V) is a ZKPOK, then there exists a witness extended emulator for (P, V).
  - Very useful when use ZKPOK inside proofs of security (and greatly simplifies)

- Can also formalize an ideal ZK functionality:
$$\mathcal{F}_{\mathrm{zk}}\big((x, w), x\big) = \big(\lambda, R(x, w)\big)$$
- **Lemma**: If (P, V) is a ZKPOK, then it securely computes the ideal ZK functionality (in the secure computation sense).

Other Applications

- A zero-knowledge proof for NQR_N
- Non-oblivious encryption
- Prove that committed value has a property, for statistically hiding
- Identification schemes

A zero-knowledge proof for $\overline{QR_N}$

Interactive proof for $\overline{QR_N}$ [GMR'85]

P $\qquad x \notin QR_N \qquad$ **V**

$$z = y^2 \quad b = 0$$

$$\longleftarrow$$

$$z = xy^2 \quad b = 1$$

$b \in_R \{0,1\}$

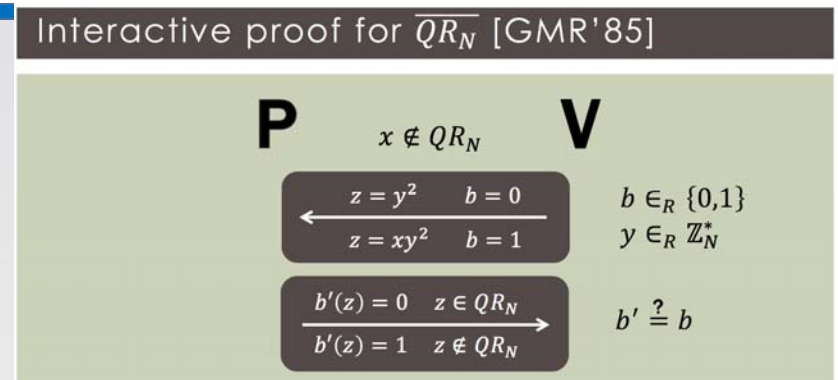
$y \in_R \mathbb{Z}_N^*$

$$b'(z) = 0 \quad z \in QR_N$$

$$\longrightarrow$$

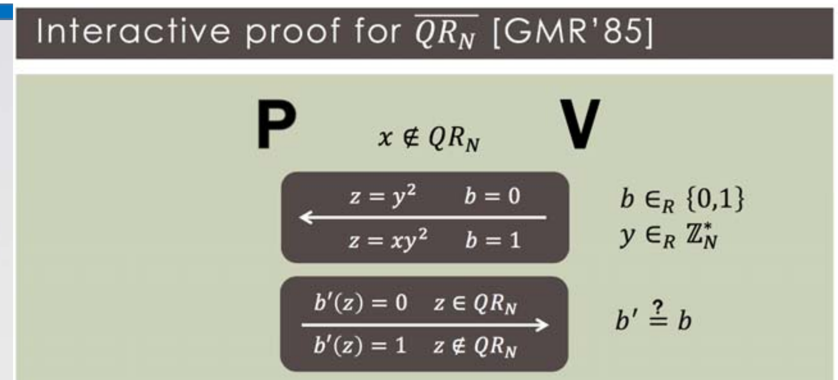
$$b'(z) = 1 \quad z \notin QR_N$$

$b' \stackrel{?}{=} b$



A ZK proof for $\overline{QR_N}$

- **Why is the proof not ZK?**
  - The verifier may have some z and wants to know if is QR or not
- **How can we make this proof ZK?**
  - The verifier sends z and proves that it **knows** y s.t. $z = xy$ or $z = xy^2$
- **Why is ZK not enough and why is a ZKPOK needed?**
  - Intuitively: for every z, there exists a y s.t. $z = xy$ or $z = xy^2$, so statement is always true
  - Formally: simulation strategy



A ZK proof for $\overline{QR_N}$

- **Simulation Strategy**
  - Simulator S runs V^* and gets z
  - Simulator doesn't know whether it should answer $b = 0$ or $b = 1$
  - Simulator runs the **knowledge extractor** on the proof from V^* and gets y
  - Simulator checks if $z = xy$ or $z = xy^2$, and so knows if $b = 0$ or $b = 1$

Non-Oblivious Encryption

- Provide an encryption and prove that you know what’s encrypted
- Motivation:
  - Prevent copying (e.g., in auction)
  - Guarantee non-malleability (did not take a previous ciphertext and maul)

Prove Property of Statistical Committed Value

- Consider a statistically-hiding commitment scheme
  - A commitment value c can be a commitment to any message
- Committer wishes to prove that it committed to a value in a certain range (or any other property)
- Statement is almost always true for any given c
- The question is whether the committer **knows** a decommitment to a message with this property
- **Rule**: whenever ZK is used with statistical hiding, ZKPOK is needed

Identification Schemes

- Alice has a public key $h = g^x$
- In order to authenticate, she proves that she knows the dlog of h

- This must be a ZKPOK, since ZK for the language of DLOG is trivial

Questions?