# Threshold Signatures

Part 2: RSA

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**Research**

The first public key signature

Let $N=pq$ be the product of two primes.

# RSA signatures

$PK=(N,e)$

$SK=d=e^{-1} \bmod \varphi(N)$

On input a message $M$, we hash it to obtain $m \in Z_N$ and compute the signature $s=m^d$

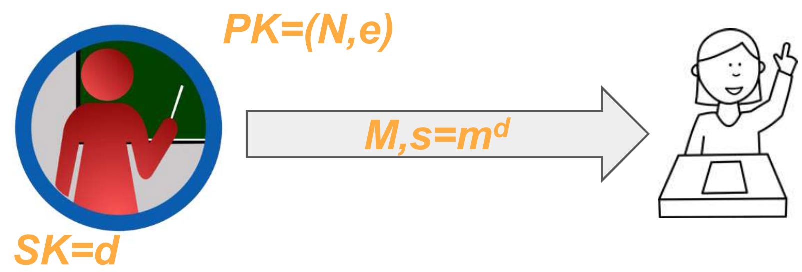
$M,s$


Computes $m=H(M)$ and $m=s^e \bmod N$

Rivest, R.; Shamir, A.; Adleman, L. (February 1978). "A Method for Obtaining Digital Signatures and Public-Key Cryptosystems". Communications of the ACM. 21 (2)

## Let’s start with additive

# *n*-out-of-*n* RSA signatures




- A ***dealer*** generates $N,e,d$ and shares the secret key $d$ among $n$ parties additively
  - Let $[d_1 \ldots d_n]$ be the shares chosen at random in $Z_{\varphi(N)}$
    - such that $d = d_1 + \ldots + d_n \bmod \varphi(N)$
  - To sign players reveal $s_i = m^{d_i} \bmod N$
    - Then $s = s_1 * \ldots * s_n \bmod N$

- Why is this secure?
  - Same interpolation in the exponent argument as in the case of dlog schemes
  - The simulator gives random $d_i$ to the adversary
    - given $s$ it can compute the partial signatures of the honest players
  - Random $d_i$ to chosen where? The simulator does not know $\varphi(N)$
    - It chooses them in $Z_N$
      - Since the uniform distributions in $Z_{\varphi(N)}$ and $Z_N$ are indistinguishable
        - When $p \sim q$

*Y.Desmedt, Y.Frankel: Threshold Cryptosystems. CRYPTO 1989: 307-315*

Move to threshold

# Shamir's over a ring

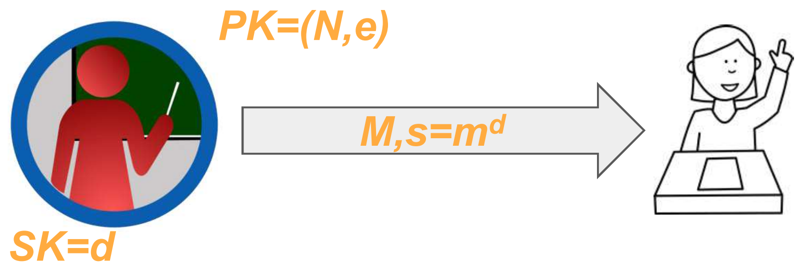
- The ***dealer*** can share $d$ with Shamir's
  - Choose a random polynomial $F(x) \in Z_{\varphi(N)}[X]$ of degree $t$ such that $F(0)=d$
  - Send to player $P_i$ the share $d_i=F(i) \bmod \varphi(N)$

- A set $S$ of $t+1$ players cannot recover the secret by polynomial interpolation
  - To compute the Lagrangians they need to invert elements $\bmod \varphi(N)$
  - Which is secret and cannot be leaked to the participants

- Remember that $d= \sum_{i \in S} \lambda_{i,S}\, d_i$
  - where $\lambda_{i,S} = [ \prod_{j \in S, j \neq i} j ] / [ \prod_{j \in S, j \neq i} (j-i)] \bmod \varphi(N)$
    - which cannot be computed by the players
- What the players can compute is $(n!)d$ by revealing $(n!)d_i$
  - Since $(n!)\lambda_{i,S}$ is an integer

## Threshold RSA First Attempt

# $t$-out-of-$n$ RSA signatures




- A ***dealer*** generates $N,e,d$ and shares the secret key $d$ among $n$ parties with Shamir $\bmod \varphi(N)$
  - Let $[d_1 \ldots d_n]$ be the shares
  - To sign players reveal $s_i = m^{d_i} \bmod N$
    - Then $s^{n!} = \prod_{i \in S} s_i^{n! * \lambda_{i,S}} = m^{d * n!} \bmod N$
- How do we get $s$?
  - Assume that $GCD(e,n!)=1$ (choose $e>n$)
    - Use Extended Euclidean algorithm to compute $a,b$ such that $a*e+b*n!=1$
  - Then by the famous Shamir's trick
    - $s = m^d = m^{d(a*e+b*n!)} = m^a * m^{b*d*n!} = m^a * s^b \bmod N$

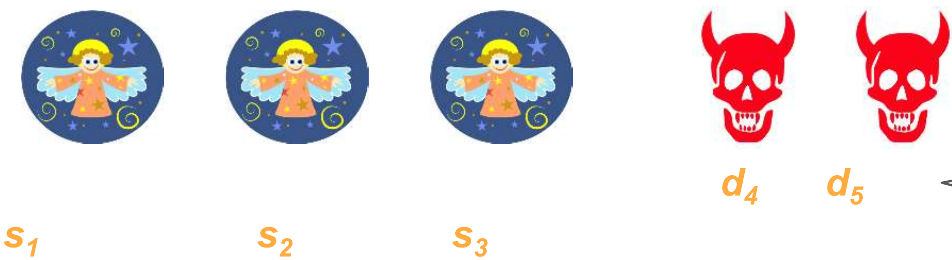
*Victor Shoup: Practical Threshold Signatures. EUROCRYPT 2000: 207-220*

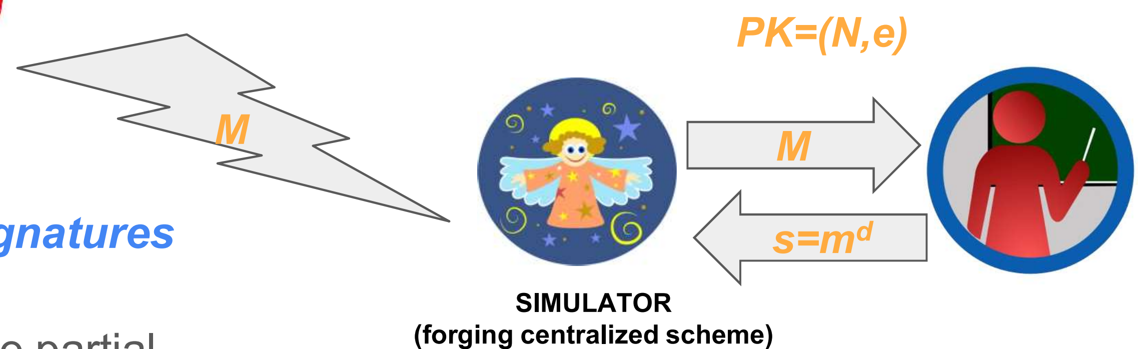
## Threshold RSA

# Let's try to Simulate

Simulator gives random $d_i$ to the adversary and plays the role of the honest players

Assume the adversary can forge controlling only $t$ players



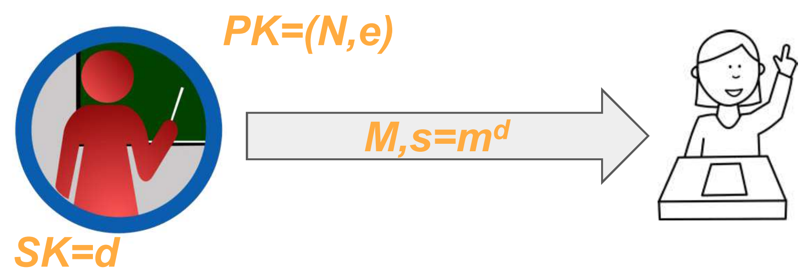




Simulator computes the adversary ***t partial signatures*** $s_i = m^{d_i}$ and knows $s_0 = s = m^d$

- But cannot interpolate in the exponent the partial signatures of the honest players
  - Since $d_j = \sum_{i \in S} \lambda_{j,i,S}\, d_i$ then $s_j = \prod_{i \in S,} s_i^{\lambda_{j,i,S}}$
  - And the Lagrangians are fractions
- He can however interpolate $s_j^{n!} = \prod_{i \in S,} s_i^{n! * \lambda_{j,i,S}}$

## Threshold RSA

# $t$-out-of-$n$ RSA signatures




- A ***dealer*** generates $N,e,d$ and shares the secret key $d$ among $n$ parties with Shamir $\bmod \varphi(N)$
  - Let $[d_1 \dots d_n]$ be the shares
  - To sign players reveal $s_i = m^{d_i * n!} \bmod N$
    - Then $s^z = \prod_{i \in S} s_i^{n! * \lambda_{i,S}} = m^{d*z} \bmod N$
    - Where $z = (n!)^2$
  - We get $s$ via the GCD trick again assuming that $GCD(e,z)=1$ (choose $e>n$)

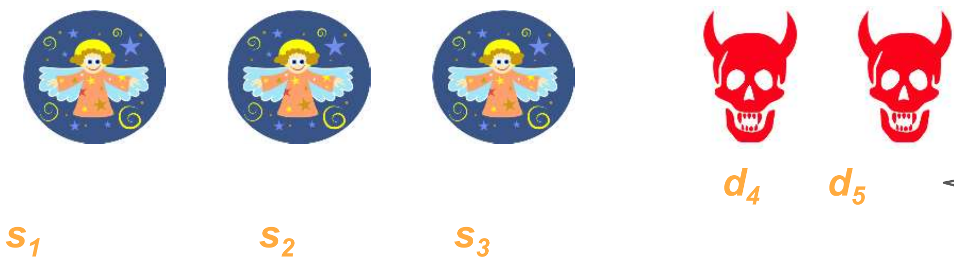
*Victor Shoup: Practical Threshold Signatures. EUROCRYPT 2000: 207-220*

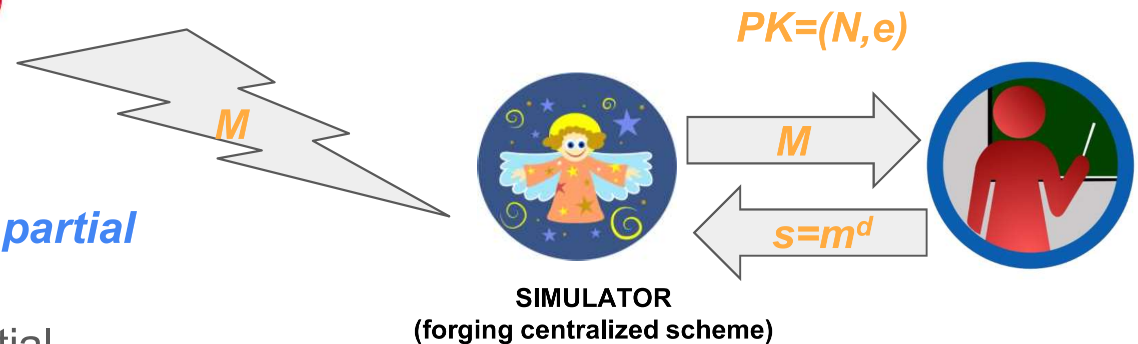
## Threshold RSA

# Simulation

Simulator gives random $d_i$ to the adversary and plays the role of the honest players

Assume the adversary can forge controlling only $t$ players



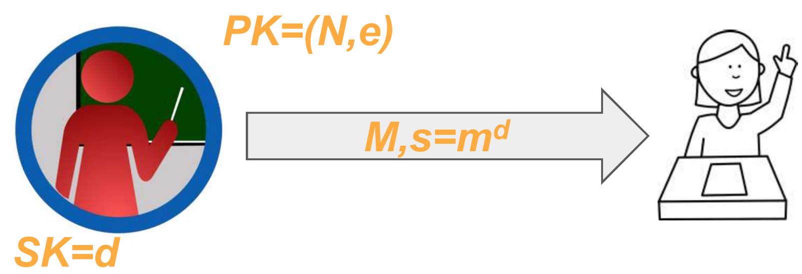




Simulator computes the adversary $t$ ***modified partial signatures*** $u_i = m^{d_i}$ and knows $u_0 = s = m^d$

- It can interpolate in the exponent the partial signatures of the honest players
  - Since $d_j = \sum_{i \in S} \lambda_{j,i,S}\, d_i$ then $s_j = \prod_{i \in S} s_i^{\lambda_{j,i,S}}$
  - And the Lagrangians are fractions
- $u_j^{n!} = \prod_{i \in S} u_i^{n! * \lambda_{j,i,S}}$

## What if the identifiers are big

# Ad-hoc groups




- In the previous solution the value $n$ is a parameter to the scheme
  - Computation is linear in $n$ (exponentiate to $n!$)
  - assumes that the identifiers of the players are exactly integers between $1$ and $n$
    - $n!$ grows really large if identifiers are random $k$-bit strings

reduction ???

- To sign players reveal $s_i = m^{d_i * n!} \bmod N$
  - Then $s^z = \prod_{i \in S} s_i^{n! * \lambda_{i,S}} = m^{d*z} \bmod N$

Computation of signature

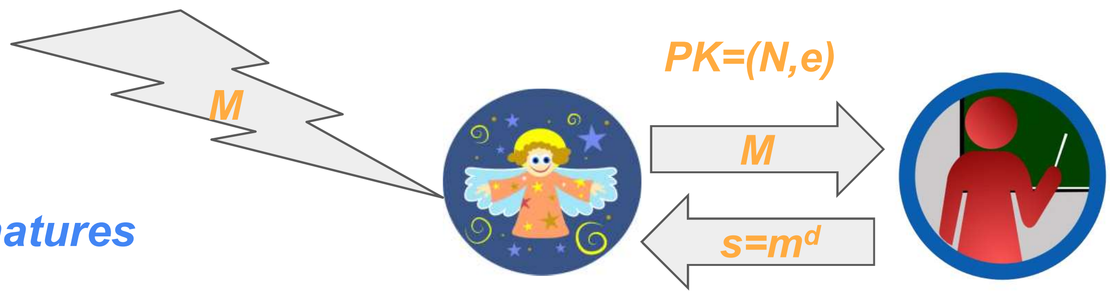
This one can be replaced with $lcm\{(i-j)\}$ for $i,j \in S$ $< 2^{kt}$

*G, S.Halevi, H.Krawczyk, T.Rabin: Threshold RSA for Dynamic and Ad-Hoc Groups. EUROCRYPT 2008*

# Back to the Simulation

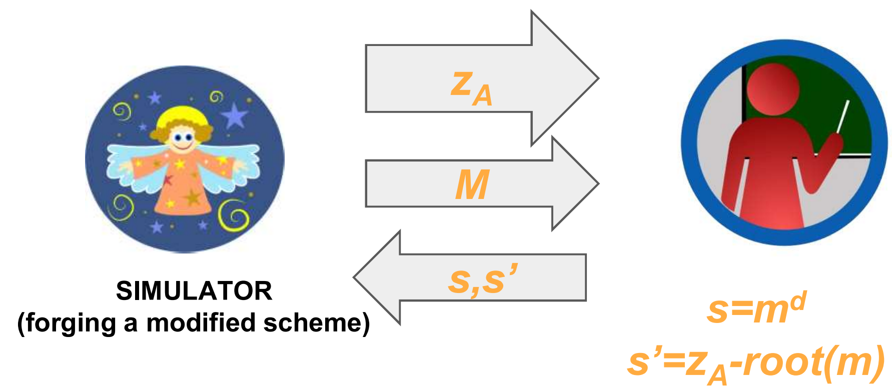
Assume the adversary can forge controlling only *t* players

Simulator gives random $d_i$ to the adversary and plays the role of the honest players


Simulator computes the adversary *t* ***partial signatures*** $s_i = m^{d_i}$ and knows $s_0 = s = m^d$

- To interpolate in the exponent the partial signatures of the honest players it has to compute a $z_A$-*root* of m
  - Where $z_A$ is the product of all the denominators of the adversary's Lagrangians




Knowledge of *s'* allows the simulator to complete the simulation

If $GCD(e,z_A)=1$ conjectured not to help find *e-roots*

# Dealing with bad partial signatures

- Remember that on message $M$ a player outputs $s_i = m^{d_i} \bmod N$
    - How to detect bad partial signatures?

- ***Message Authentication Codes:***
  - For every share $d_i$, the dealer chooses $n$ triplets $(a_{ij}, b_{ij}, c_{ij})$ such that
    - $a_{ij} * d_i + b_{ij} = c_{ij}$ ***over the integers***
    - With $a_{ij} \in [0 \ldots 2^{k1}]$ and $b_{ij} \in [0..2^{k2}]$ chosen uniformly at random
    - And sends $c_{ij}$ to player $i$ and $a_{ij}, b_{ij}$ to player $j$

  - When player $i$ outputs $s_i = m^{d_i} \bmod N$
    - It sends to player $j$ the value $C_{ij} = m^{c_{ij}} \bmod N$
    - Player $j$ accepts $s_i$ if $s_i^{a_{ij}} * m^{b_{ij}} = C_{ij} \bmod N$

*G, S.Jarecki, H.Krawczyk, T.Rabin: Robust and Efficient Sharing of RSA Functions. J. Cryptol. 13(2): 273-300 (2000)*

## Adding robustness

# Dealing with bad partial signatures

- Remember that on message ***M*** a player outputs $s_i = m^{d_i} \bmod N$

- ***Zero-Knowledge Proofs:***
  - For every share $d_i$, the dealer publishes $G_i = g^{d_i} \bmod N$

  - When player ***i*** outputs $s_i = m^{d_i} \bmod N$
    - It also sends a ZK proof that $s_i$ and $G_i$ to have the same discrete log with respect to ***m*** and ***g***
      - It requires restricting ***m,g*** to a cyclic subgroup of $Z_N^*$
        - For safe primes the subgroup of quadratic residues

*G, S.Jarecki, H.Krawczyk,T.Rabin: Robust and Efficient Sharing of RSA Functions. J. Cryptol. 13(2): 273-300 (2000)*

## Chaum's prescience

# Equality of discrete log ZK Proofs in groups of prime order

$$y=g^x \quad s=m^x$$

D.Chaum, T.P. Pedersen:
Wallet Databases with Observers. CRYPTO 1992

$\$ \; r \in Z_q$

$a=g^r, b=m^r$ →

← $c$

$\$ \; c \in Z_q$

$d=r+cx \bmod q$ →

$ay^c ?= g^d$

$bs^c ?= m^d$

Public coin: can be made non-interactive via Fiat-Shamir. Proof of knowledge of $x$

D.Chaum, H.Van Antwerpen:
Undeniable Signatures. CRYPTO 1989

← $c=g^a m^b$

$\$ \; a,b \in Z_q$

$d=c^x$ →

$d ?= y^a s^d$

Private coin (can't be made non-interactive). Two round HVZK (can be turned into 4-round full Zk)

**Composite order**

# Equality of discrete log ZK Proofs in groups of unknown order

$y=g^x \; s=m^x$

$ $r \in [0 \ldots N*B]$

$a=g^r, b=m^r$

$c$

$ $c \in [0 \ldots B']$

$d=r+cx$ over the integers

$ay^c ?= g^d$

$bs^c ?= m^d$

mod $N$

Choose r large enough to statistically hide $x$

$c=g^a m^b$

$ $a,b \in Z_N$

$d=c^x$

$d ?= y^a s^d$

G, S.Jarecki, H.Krawczyk,T.Rabin: Robust and Efficient Sharing of RSA Functions. J. Cryptol. 13(2): 273-300 (2000)

## Wait a minute

# DEALER?

- This time removing the dealer is not as easy as in the case of discrete log based schemes
  - The dealer does not just generate a random value
  - It generates an RSA modulus ***N*** with its factorization and then the values ***e,d***

- To replace the dealer we need to come up with a protocol to do all of the above distributed with the above secrets (the factorization) in shared form
  - While in principle this is obtainable via MPC protocols it is still a difficult task to perform efficiently
    - The bottleneck would be the repeated computation of modular exponentiations in a distributed Miller-Rabin primality test
    - This has been a very active research area

Dan Boneh, Matthew K. Franklin: Efficient generation of shared RSA keys. J. ACM 48(4): 702-722 (2001)

# Avoiding Miller-Rabin

- Let's break down the task:
  a. The ***n*** parties generate a random number and do a preliminary sieving (to make sure that it is not divided by small primes)
  b. Given two such numbers ***p,q*** the parties compute ***N=pq***
  c. The parties now distributively test that ***N*** is bi-prime (the product of 2 primes)
  d. If the test succeeds the parties compute ***e,d***

Dan Boneh, Matthew K. Franklin: Efficient generation of shared RSA keys. J. ACM 48(4): 702-722 (2001)

# Sieving and Multiplication

a. The n parties generate a random number and do a preliminary sieving (to make sure that it is not divided by small primes)
   - Each party generate random numbers $p_i \in [0 \ldots B]$ and $r_i \in [0 \ldots B']$
   - Reconstruct ***pr***
     - Multiplication of additively shared values
     - And reject ***p*** if ***pr=0 mod a***
       - Where ***a*** is the small prime

a. Given two such numbers ***p,q*** the parties compute ***N=pq***
   - Again this is the multiplication of additively shared values

Dan Boneh, Matthew K. Franklin: Efficient generation of shared RSA keys. J. ACM 48(4): 702-722 (2001)

# Bi-primality testing

c. The parties now distributively test that $N$ is bi-prime (the product of 2 primes)
   - A very simplified version
   - Remember that $N=pq$ and the parties have additive sharings of $p,q$
   - If $N$ is bi-prime then $\varphi(N)$ is the order of $Z_N^*$
     - The parties have an additive sharing $\varphi_1 \ldots \varphi_N$ of $\varphi(N)=N-p-q+1$
   - ***Repeat many times***:
     - The parties choose a random value $g$ and test if $g^{\varphi(N)}=1$
     - ***Locally*** compute $g_i=g^{\varphi i}$
     - Use a distributed computation to check that $g_1 * \ldots * g_n = 1$
       - Can't reveal the $g_i$
   - An additional GCD test is also required

D.Catalano, G, S. Halevi: Computing Inverses over a Shared Secret Modulus. EUROCRYPT 2000: 190-206

# Inversion over a shared secret

d. The parties now choose $e$ and compute $d=e^{-1} \bmod \varphi(N)$
   - This is the "dual" problem of the one we saw yesterday
     - In the DSA scheme we had a public modulus and we had to invert the secret
     - Here we have a public value to invert but a secret modulus
   - The parties have an additive sharing $\varphi_1 \ldots \varphi_N$ of $\varphi(N)$
     - Choose a random value $r_i \in [0 \ldots B]$ and let $r = r_1 + \ldots + r_n$
     - Reveal $a_i = \varphi_i + er_i$
       - $a = a_1 + \ldots + a_n = \varphi(N) + re$
     - If GCD(a,e)=1 then there exists b,c such that ab+ce=1
       - $1 = ab + ce = b\varphi(N) + (br+c)e$
       - $br+c = e^{-1} \bmod \varphi(N)$
     - Shares of $d$ can be easily obtained by setting $d_i = br_i$
       - With one party adding $c$ as well

D.Catalano, G, S. Halevi: Computing Inverses over a Shared Secret Modulus. EUROCRYPT 2000: 190-206

# Signatures based on Strong-RSA

We have been looking at the basic "hash and sign" RSA signature

- Which are proven in the random oracle model

- There are provably secure schemes based on the Strong-RSA assumption
  - Given $(N,g)$ find $(e,s)$ such that $s^e=g \bmod N$
- These schemes work as follows:
  - The public key is $(N,g)$ and the secret key is $\varphi(N)$
  - a message M is mapped into an exponent m and the signature is $s=g^d \bmod N$ where $d=m^{-1} \bmod \varphi(N)$
  - The pair $(M,s)$ is valid if $s^m=g \bmod N$
    - G, S.Halevi, T.Rabin: Secure Hash-and-Sign Signatures Without the Random Oracle. EUROCRYPT 1999: 123-139
    - R.Cramer, V.Shoup: Signature schemes based on the strong RSA assumption. ACM Trans. Inf. Syst. Secur. 3(3): 161-185 (2000)

- To make these schemes into threshold ones we need exactly the protocol we showed before
  - Given $m$ compute a sharing of $d=m^{-1} \bmod \varphi(N)$
  - Over a distributed $\varphi(N)$

N.Gilboa:Two Party RSA Key Generation. CRYPTO 1999: 116-129

# The two-party case

The Boneh-Franklin protocol required honest majority and was proven only for the honest but curious adversary setting

- Gilboa showed how to extend it for the 2-party case
- In particular introducing the MtA protocols we discussed yesterday

M.Chen, C.Hazay, Y.Ishai, Y.Kashnikov, D.Micciancio, T.Riviere, a.shelat, M.Venkitasubramaniam, R.Wang:Diogenes: Lightweight Scalable RSA Modulus Generation with a Dishonest Majority. IEEE Symposium on Security and Privacy 2021: 590-607

# Many follow up works

There are several applications beyond threshold RSA signatures that could use a distributed generation of RSA moduli

- Many protocols have been presented following the Boneh-Franklin approach with improvements focused on
  - Increasing the rate of sieving to avoid running the bi-primality test too often
  - Reducing communication complexity
    - E.g. use a distributed version of the MtA protocol using a threshold additively homomorphic encryption
    - Since one cannot use Paillier, use lattice-based one instead
  - Adding security against malicious adversary via ZK proofs
    - Using recent advances in SNARKs (sublinear size proofs)
  - We can now generate distributed RSA moduli for 1000’s of parties in a matter of minutes.