Private Set Intersection (PSI)

Malicious security, and amplifying the success probability

Benny Pinkas, Bar-Ilan University

In this lecture

- Malicious security for PSI
- Amplifying the success probability
- PSI conclusions

(many slides by my coauthors)

Template for PSI based on OPRF (previous hour)

$x_1,\ldots,x_n$ 

 $y_1,\ldots,y_n$

K is a OPRF key known to Bob

$x_1,\ldots,x_n \rightarrow$ **OPRF** $\leftarrow$ K

$\boxed{F_K(x_1),\ldots, F_K(x_n)}$ $\leftarrow$

$\leftarrow$ $\boxed{F_K(y_1),\ldots, F_K(y_n)}$ PRF of Bob's inputs

Compares the two lists

Implementing the OPRF

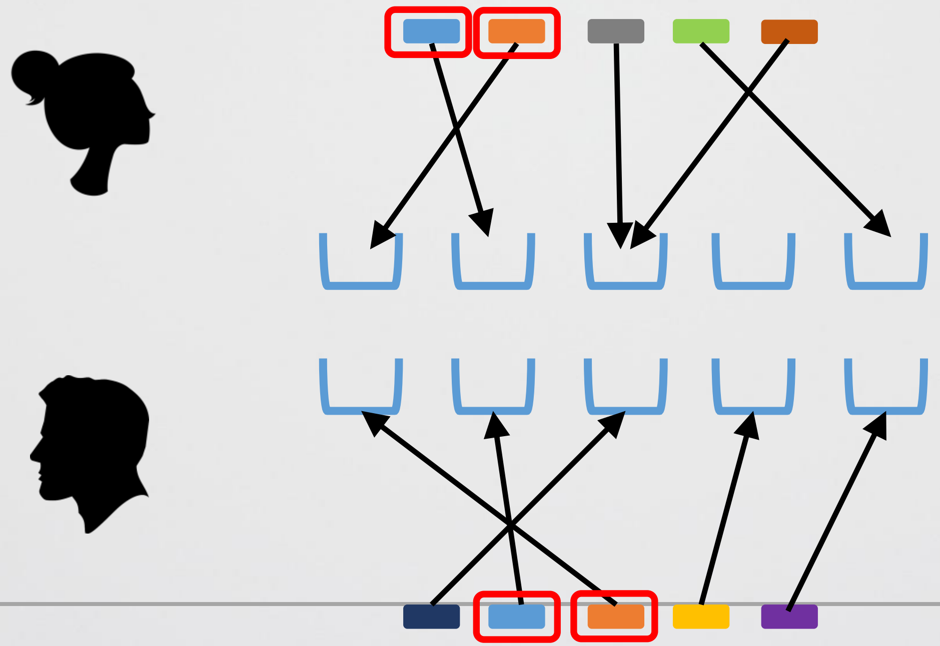
- The most efficient OPRF implementations are based on OT extension

- Caveat: Secure only as long as client evaluates the OPRF <u>at most once</u>
- E.g., when $F_{a,b}(x) = ax+b$

Solution: Hashing

- Suppose both parties use the same public random hash function h() to hash their n items to n bins.
  - Then obviously if Alice and Bob have the same item, both of them map it to the same bin. PSI can then be independently run for each bin.
  - If Bob has a single item in each bin, he only needs to evaluate the OPRF once

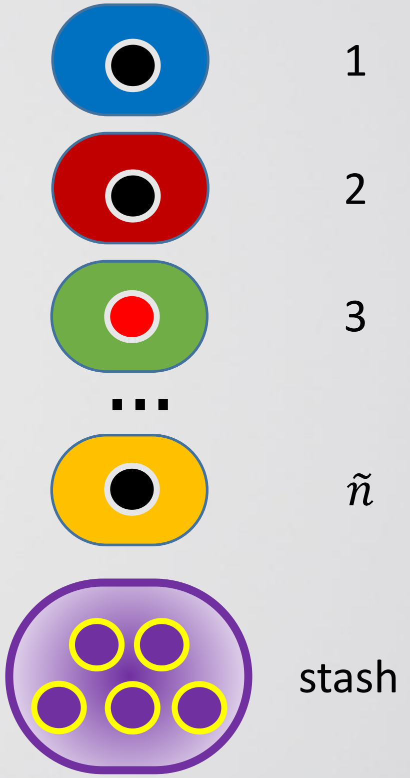
Problem: many bins will have >1 item mapped to them



Using 2 Hash Functions (cuckoo hashing [PR,KMW])

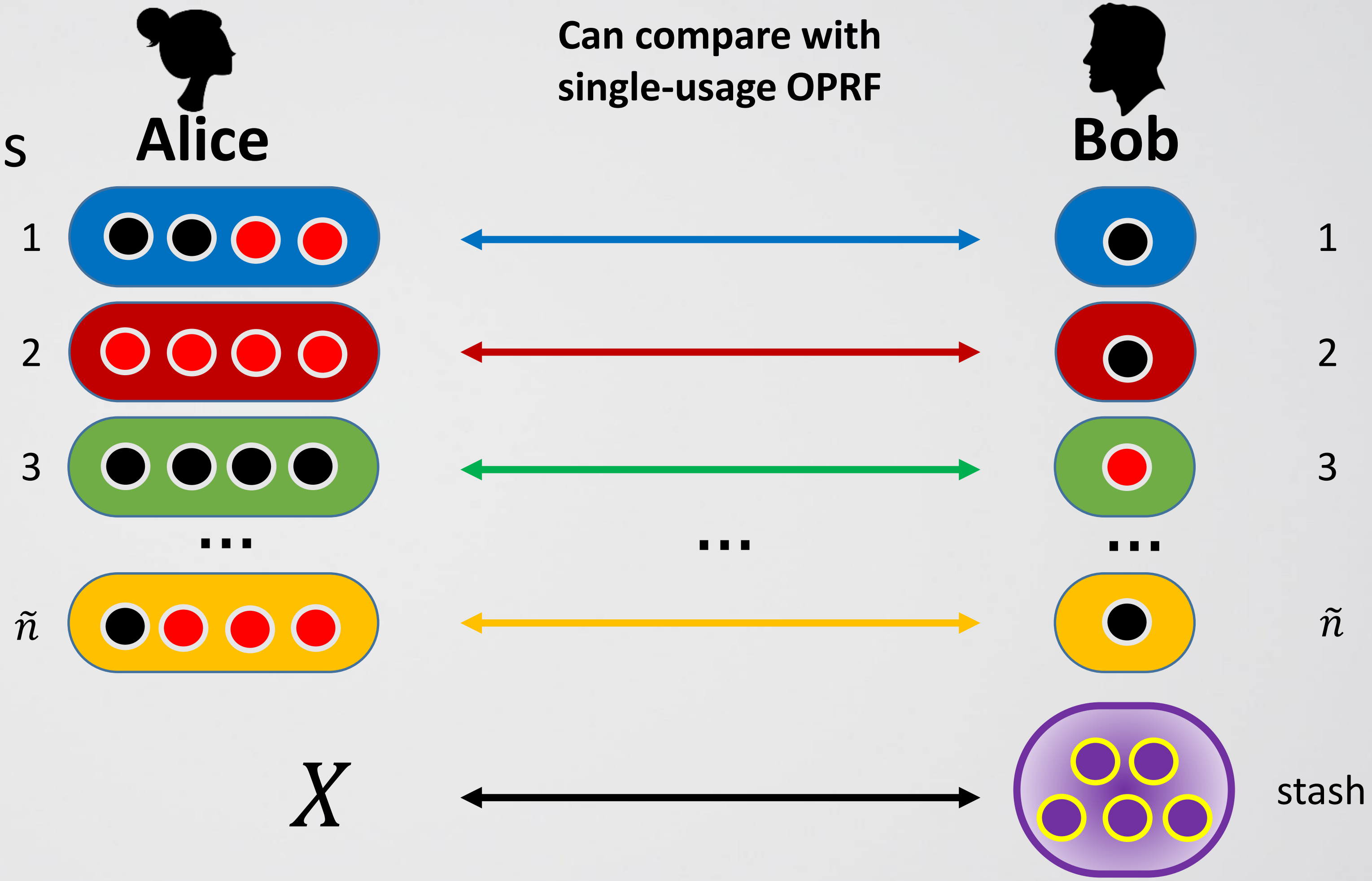
- h_1, h_2: item $\rightarrow$ bin
- Map n items to $(2+\epsilon)n$ bins
- Each bin stores at most one item!

- Succeeds with very high probability
- If we also have a stash of size s, all items *x* can be mapped to either $h_1(x), h_2(x)$ or the stash, except with probability $O(n^{-(s+1)})$



The Power of Using 2 Hash Functions (Cuckoo)

- h_1, h_2: item $\rightarrow$ bin
- Map n items to $(2+\epsilon)n$ bins
  - Alice – **simple** hashing
    - $x \rightarrow h_1(x)$ **and** $h_2(x)$
  - Bob – **Cuckoo** hashing
    - $y \rightarrow h_1(y)$, **or** $h_2(y)$
- **Caveat:** stash size is $\omega(1)$ (let's ignore it)



Combining cuckoo hashing with PSI

- In each bin, Bob (who uses CH) has one item *y*. Alice has O(log n) items $x_1, x_2, \ldots$
- In each bin, they run an OT-based OPRF of a function *F*, so that Bob learns *F(y)*, and Alice can compute *F()* on any input
- Alice sends to Bob the *F()* values she learned in all bins
- Bob compares them to the values that he learned

Why isn't this secure against malicious parties?

It turned out that only the following attack is problematic:

- Suppose that both Alice and Bob have a value z
  - Alice should put z in bins $h_1(z)$ and $h_2(z)$
  - Bob uses CH and puts z in only one of these two bins
- Suppose Alice chooses to put z only in bin $h_1(z)$
- Then z will be in the PSI output **iff** Bob chose to put z in $h_1(z)$
- This decision of Bob depends on the **other** inputs that he has
- → The output of the PSI leaks information about **other** inputs of Bob

The function F() that is used

- F() can be implemented using oblivious transfer extension
- Specifically, a protocol of Orru, Orsini and Scholl, uses OT-extension to implement F(), with the following properties
  - The construction is secure against malicious behavior
  - For each table entry i, the receiver learns $F_i(x)$, and the sender can compute $F_i()$
  - Important for this lecture: A **homomorphic property**: $F_i(x) + F_j(y) = F_{i+j}(x \oplus y)$.

PSI from PaXoS, OKVS, and amplification

Relevant papers:

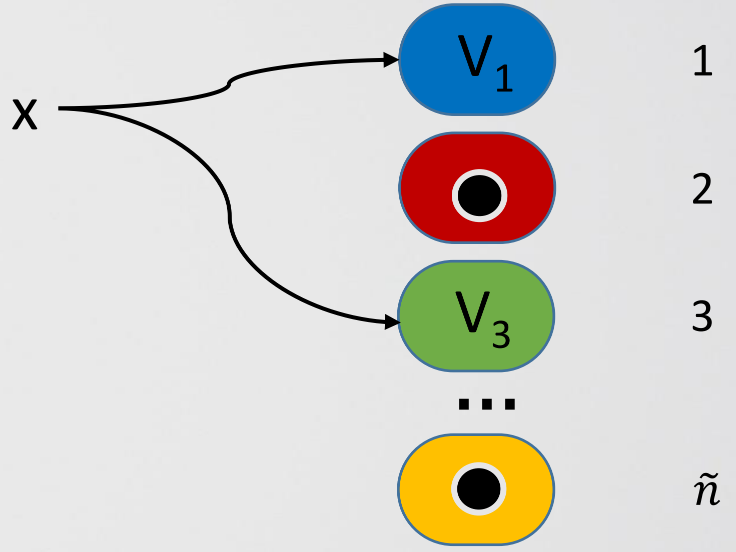
- **Malicious security for OT extension: “PSI from OT”.** Actively Secure 1-out-of-N OT Extension with Application to Private Set Intersection. Michele Orrù and Emmanuela Orsini and Peter Scholl. (CT-RSA 2017)

- **Efficient malicious PSI: PSI from PaXoS:** Fast, Malicious Private Set Intersection. Benny Pinkas and Mike Rosulek and Ni Trieu and Avishay Yanai (Eurocrypt 2020)

- **Even more efficient malicious PSI; amplification of success probability:** Oblivious Key-Value Stores and Amplification for Private Set Intersection. Gayathri Garimella and Benny Pinkas and Mike Rosulek and Ni Trieu and Avishay Yanai. (Crypto 2021)

A different flavor of cuckoo hashing

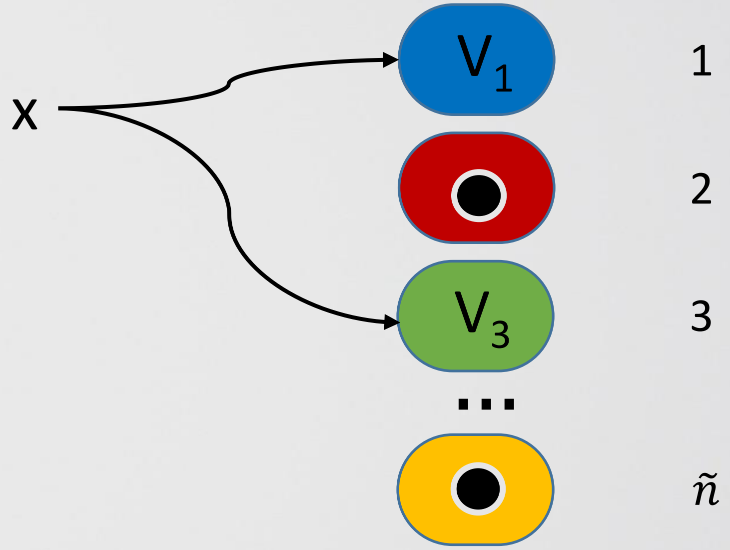
- Bob is using CH
- Suppose that x is mapped to locations $h_1(x)=i$ and $h_2(x)=j$.
- Unlike CH, Bob puts there values V_i and V_j such that $V_i \oplus V_j = x$
- Suppose that this mapping is possible, and this property holds for all items that Bob inserts (this is similar to a garbled Bloom filters)



$X = V_1 + V_3$

A different flavor of cuckoo hashing

- In the PSI protocol, Bob runs the OPRF in the bins so that he learns $F_i(V_i)$ and $F_j(V_j)$
- Recall the homomorphic property of the function: $F_i(x) + F_j(y) = F_{i+j}(x \oplus y)$
- Therefore Bob can compute $F_i(V_i) + F_j(V_j) = F_{i+j}(V_i \oplus V_j) = F_{i+j}(x)$
- Alice sends to Bob, for each input y of her, $F_{h1(y)+h2(y)}(y)$
- Security: Alice cannot cheat by sending just one of $F_{h1(y)}(y)$, $F_{h2(y)}(y)$ (this needs a proof)



$X = V_1 + V_3$

OKVS Example – Encoding in PaXoS (simplified*)

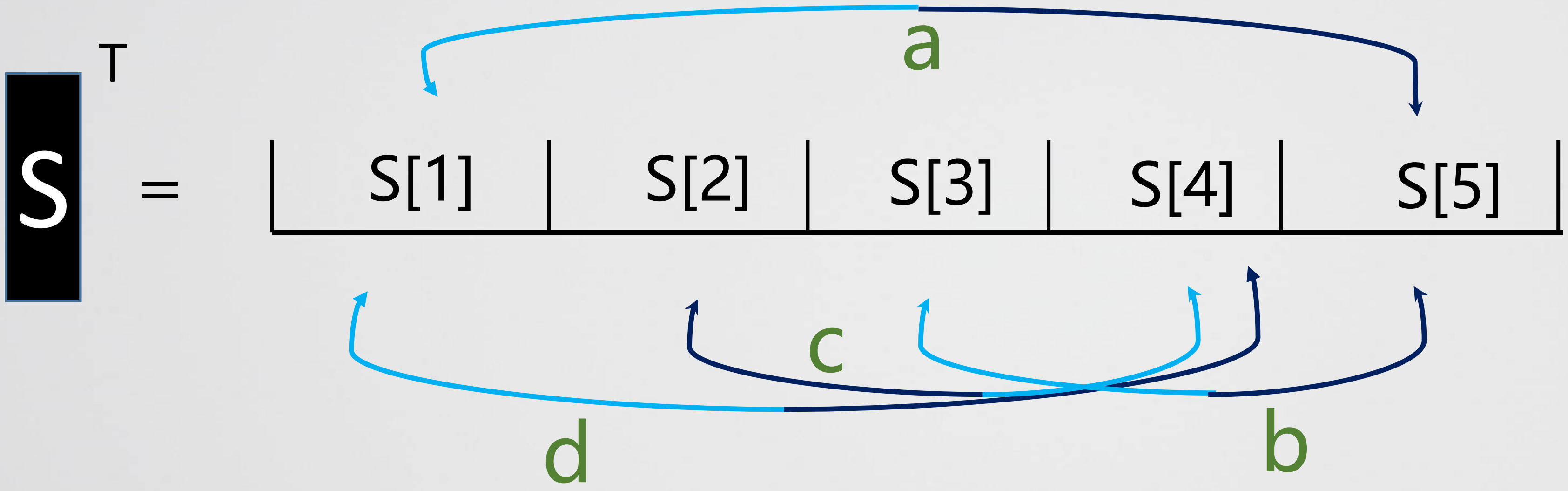
key-value <a, val(a)>

Decode(a) = S[1] ⊕ S[5] = val(a)

How do we encode many such keys such that they decode correctly?

OKVS Example – Encoding in PaXoS (simplified*)

Encode keys: a, b, c, d



Peeling:

c : slot S[2]
d : slot S[4]
b : slot S[3]
a : slot S[1], S[5]

Solving for 'S' :

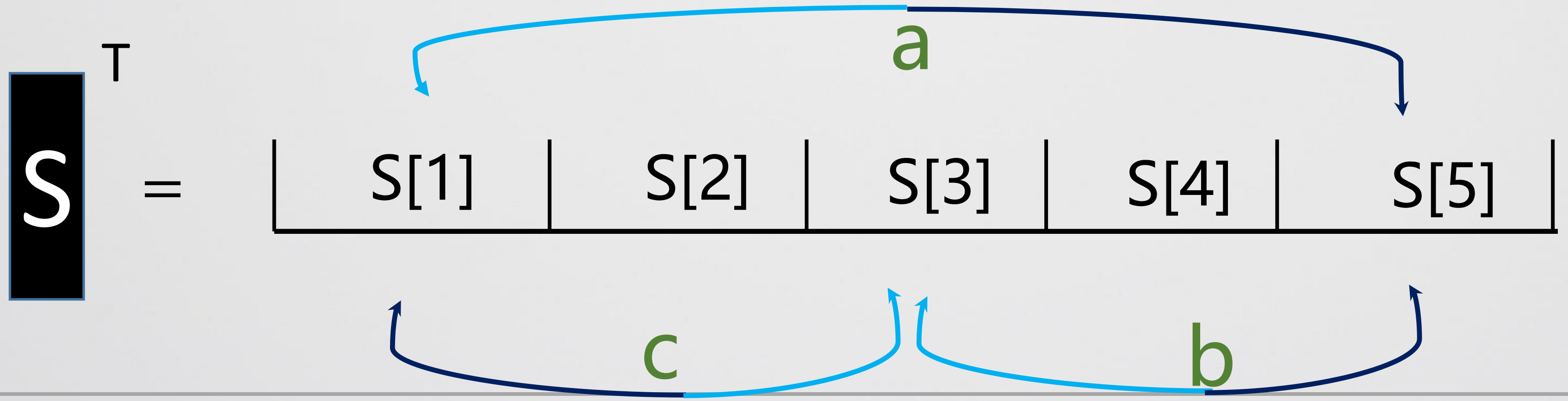
$S[1] \oplus S[5] = \mathrm{val}(a)$
$S[3] = S[5] + \mathrm{val}(b)$
$S[4] = S[1] + \mathrm{val}(d)$
$S[2] = S[4] + \mathrm{val}(c)$

recursively find slots constrained by just one key

Does encoding always work?

What happens when peeling fails?

- The **2-core** of a graph is the maximum subgraph where each node has degree at least 2
  - Namely, the subgraph containing all cycles, as well as all paths connecting cycles.
- All values (edges) which are **not** in 2-core can be handled via peeling
  - But, peeling does not work on the 2-core



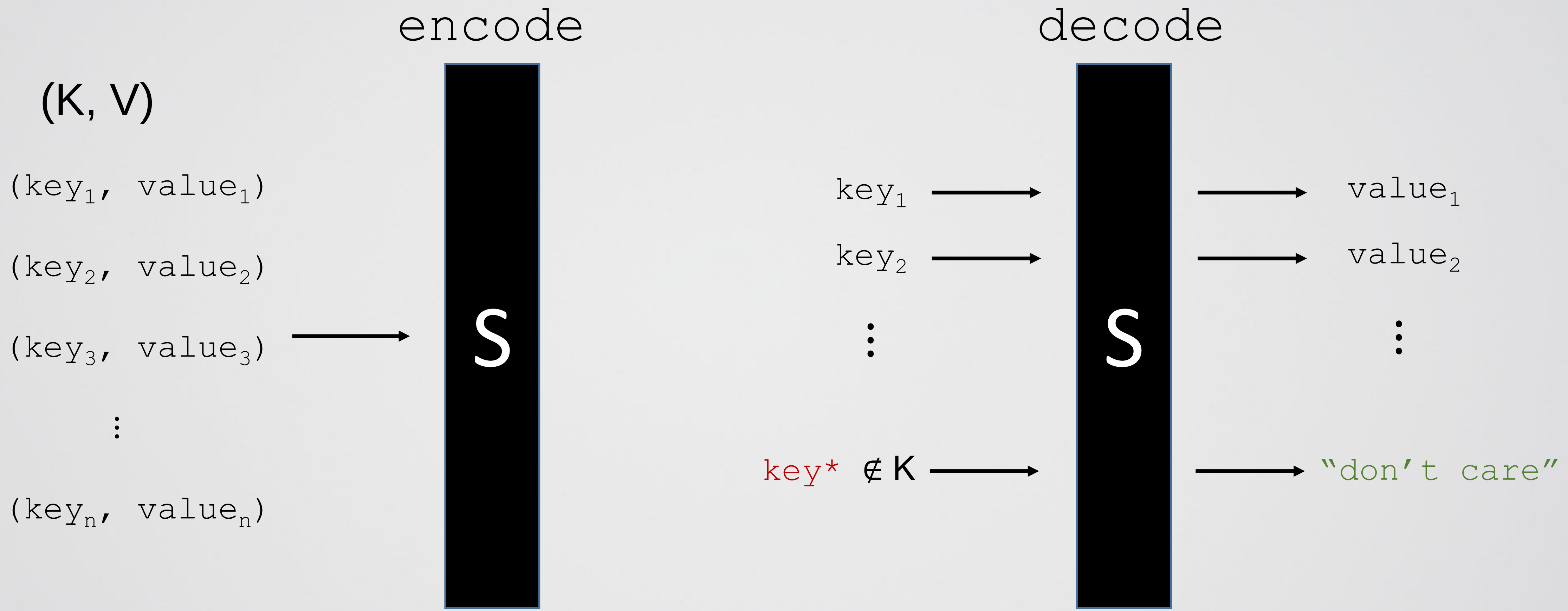
What happens when peeling fails?

- THM*: For a CH graph of size O(n), WHP the 2-core of size O(logn) ☺
- In other words, the encoding the we suggested can handle all but O(logn) of the items mapped to the CH
  - Handling only O(logn) items should be efficient
  - But we must hide which items these are

What do we actually need?

- An “Oblivious Key-Value Store” (OKVS)
- Key-Value Store:
  - Encode a set of (key, value) pairs. Querying an encoded key returns the corresponding value.
- Oblivious Key-Value Store (OKVS):
  - Hide the keys!
    - A query for an encoded key k will return the corresponding value
    - A query for a key which is not encoded will return a random value
    - Suppose all encoded values in (key,value) pairs are random
    - These two options will be indistinguishable for those making the queries
- This is a recurring requirement in PSI protocols [CDJ16,KMP+17,PSTY19, PRTY19, KRTW19]…

Oblivious Key-Value Store (OKVS)



if values are random; then S hides encoded keys; hence oblivious

Properties of OKVS

$(\mathrm{K}, \mathrm{V}) = (k_1, v_1), (k_2, v_2), \dots, (k_n, v_n)$

$$S^T = \begin{array}{|c|c|c|c|} S[1] & S[2] & \cdots & S[m] \end{array}$$

A must for the PSI constructions we saw

Linear OKVS : if values are in $\mathcal{F}$, use decoding function $d : K \rightarrow \mathcal{F}^m$

Binary OKVS : special case where $d(k) = \{0, 1\}^m$

$$\underbrace{\begin{bmatrix} - & d(k_1) & - \\ \vdots & \ddots & \vdots \\ - & d(kn) & - \end{bmatrix}}_{n \times m} \times \begin{matrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{matrix} = \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix}$$

OKVS efficiency measures

Size: $\frac{n}{m}$ (optimal = 1)
Encoding time : solve for 'S'
Decoding time : matrix mult

OKVS Examples - PaXoS

OKVS efficiency measures for PaXoS

- The memory is linear in n
- Encoding time and decoding time are linear
- But cannot encode all items – failure for those which happen to be in the 2-core

OKVS Examples - Polynomial

$$(\mathrm{K},\mathrm{V}) = (k_1, v_1), (k_2, v_2), \dots, (k_n, v_n) \xrightarrow{\text{encode}} S(x) = s_1 + s_2 x^1 + s_3 x^2 + \dots\ s_n x^{n-1}$$

solve for 'S'

$$\begin{bmatrix} 1 & k_1 & k_1^2 & k_1^3 & k_1^3 & k_1^4 \\ & \ddots & & & \ddots & \\ \vdots & & & & & \vdots \\ 1 & k_6 & k_6^2 & k_6^3 & k_6^4 & k_6^5 \end{bmatrix} \times \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{matrix} = \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{matrix}$$

$$S^T = [s_1\ s_2\ s_3\ \dots\ s_n]$$

OKVS efficiency measures
Linear (optimal) size
Encoding time and Decoding time
= O(n log² n) field operations (FFT)

OKVS Examples – Random Matrix

Pick a random matrix of size $(n \times m)$ of field elements (row corresponding to key is defined as H(key))

$$\underbrace{\begin{bmatrix} r_1 & r_2 & r_3 & r_4 & & r_m \\ & \ddots & & & \ddots & \\ \vdots & & & & & \vdots \\ & & & & & \end{bmatrix}}_{n \times m} \times \begin{matrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ \\ s_m \end{matrix} = \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \\ v_n \end{matrix}$$

OKVS efficiency measures

Size is linear
Encoding time = $O(n^3)$
Decoding time = $O(n^2)$

Pr[Bad event: random matrix has linearly dependent rows] < $|\mathcal{F}|^{n-m-1}$
Binary OKVS : $d(k) = \{0, 1\}^m$ need $m \geq n + \lambda - 1$ for error probability $2^{-\lambda}$

Handling the 2-core in PaXoS

- Suppose I know in advance that whp $|\text{2-core}| = O(\log n)$
- We can encode these logn items using a less efficient OKVS, e.g. a random matrix
- Advantage: This requires only $\log n + \lambda$ variables to encode logn values. Total OKCS size is $O(n) + O(\log n) + \lambda$
- Encoding takes $(\log n + \lambda)^3$ time, but this is fine.

The full solution (read on your own)

- The parties agree on adding $O(\log n) + \lambda$ variables, and a random mapping H() to subsets of these variables.
- The value of each input x is defined as the **sum** of the values of the two locations to which it is mapped in the CH, and the random subset H(x) of the additional variables to which it is mapped.
- Bob maps his n inputs to a CH of size $(2+\epsilon)n$
- Bob does peeling and remains with a 2-core of size O(logn)
- Bob sets random values to the nodes in the 2-core, but solves equations with the remaining $O(\log n) + \lambda$ variables, to ensure that the values of items in the 2-core is correct.
- Bob reverses the peeling to set values to nodes, ensuring the right values to all remaining variables.
- Bob uses the OPRF to learn a value from each bin, sums them up, and learns F(x) for all inputs.

Expensive: $O(\log n + \lambda)^3$

Cheap: $O(n)$

What are concrete parameters for OKVS?

Theorem: If $\Pr[|\text{2-CORE}| \geq O(\log n)] \leq \epsilon$; $|S| = O(n) + O(\log n) + \lambda$
we can encode successfully with negligible error $\epsilon + 2^{-\lambda}$

PaXoS[PRTY20]: Table size |S| = 2.4n (heuristic), Pr[Encoding Fails] = 2^{-40}

The elephant in the room (for many PSI results): Rigorous analysis to translate the asymptotic theorem to concrete parameters ??

What are concrete parameters for OKVS?

Theorem: If $\Pr[|\text{2-CORE}| \geq O(\log n)] \leq \epsilon$; $|S| = O(n) + O(\log n) + \lambda$
we can encode successfully with negligible error $\epsilon + 2^{-\lambda}$

[Wal21a] **3**-cuckoo hash → |S| = 1.23n (empirically extrapolated)

Empirical confidence? How can we claim, with 0.9999 confidence, that

Except with probability 2^{-40} can we encode using 3-cuckoo hashing

"1 M keys into 1.3 M bins with less than 10 keys in 2-CORE" ?

Simulation is very resource intensive!!

What if the application needs failure probability 2^{-80} ?

Using probabilistic constructions for PSI

- Hashing is a probabilistic process
  - Sometimes it fails. In systems this results in higher overhead (not a big deal).
- For PSI, a hashing failure results in either
  - Inaccurate output (based on a subset of the original input set), or
  - Information leakage
- For some applications this does not matter much
  - ML ?
  - CSAM detection (false negatives are fine)

Using probabilistic constructions for PSI

- For a theoretical analysis, we want a negligible failure probability (smaller than any polynomial function)
- For a concrete analysis we want the failure probability to be, e.g., $< 2^{-40}$
- Typically, cuckoo hashing constructions have a very sharp threshold ☺
  - E.g., cuckoo hashing with 3 hash functions succeeds when |Table| > 1.23n
- But there is no tight analysis of the failure probability ☹
  - E.g., if the table is of size 1.3n, what's the probability of failure?
- Solutions?

  Heuristics; experiments (costly); amplification of success probability.

Using probabilistic constructions for PSI

Things to note:

- Typically, cuckoo hashing constructions have a very sharp threshold
  - So, in practice, by using a slightly larger hash table, hashing should work well

- The failure probability is a function of the input size
  - For small inputs, failure probability might be too large ☹
  - E.g., a failure probability of **O**(n^{-3}) (what constants?) might not be sufficiently small when $n=1{,}000$

New approach: amplification

We can very efficiently verify statements about large failure probabilities:

E.g., that with 0.9999 confidence, it holds that 3-cuckoo hashing can encode

"1k keys into 1.3k bins/slots with less than 10 keys in 2-CORE"

with failure probability $< 2^{-15}$

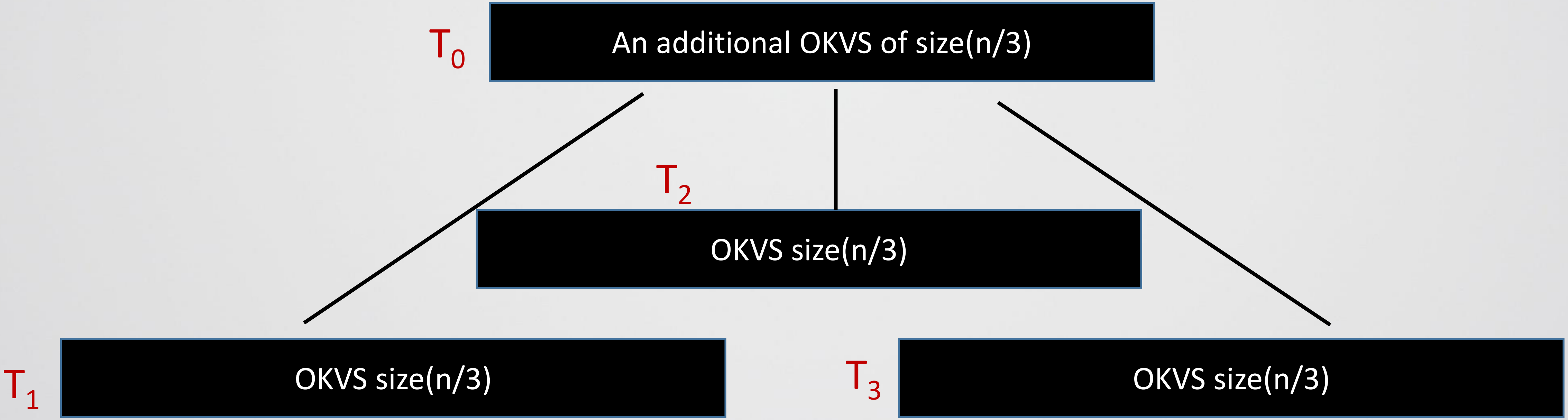
Main idea

Compose empirically verified "smaller" OKVS instances into "larger" OKVS

provably amplifying the correctness guarantee from 2^{-15} to say, 2^{-40}

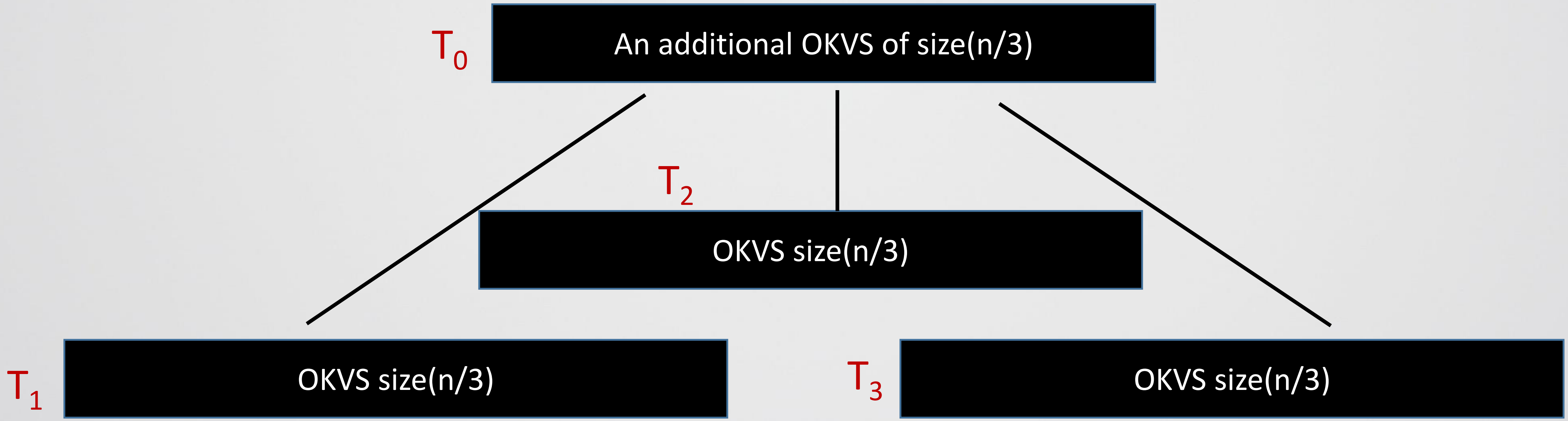
Star architecture

- 4 OKVS instances, each large enough to encode n/3 items, with failure probability p
- A hash function H() which maps items to one of T_1,T_2,T_3.



Star architecture - decoding

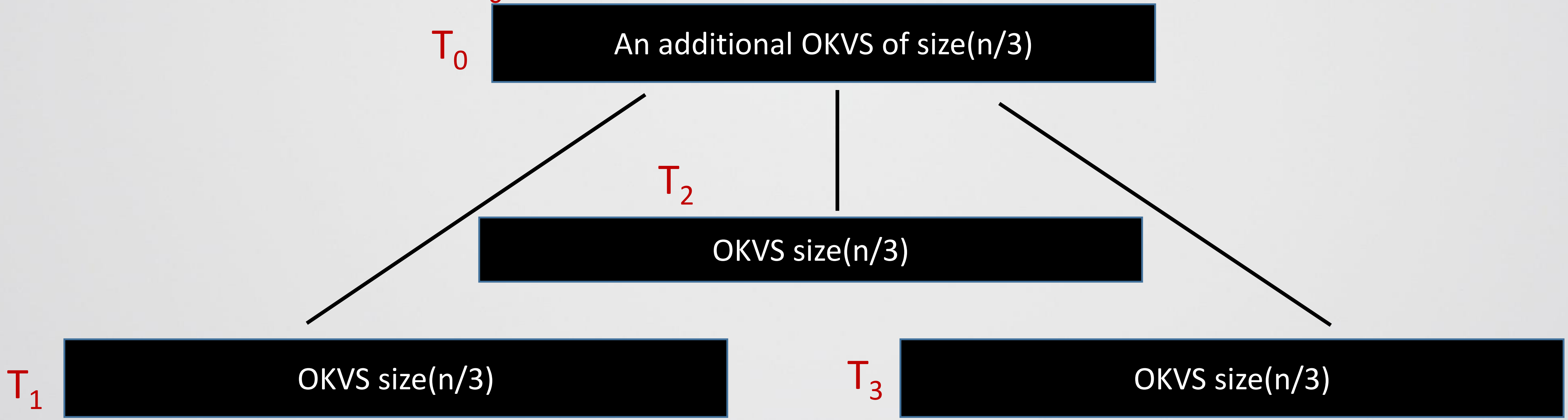
- Given a key k, read its values from tables T_0 and $T_{H(k)}$ and return the XOR of these results: Decode(k) = Decode(T_0,k) XOR Decode($T_{H(k)}$,k)



Star architecture - encoding

(The success of encoding into a table is a function of the keys, not the values)

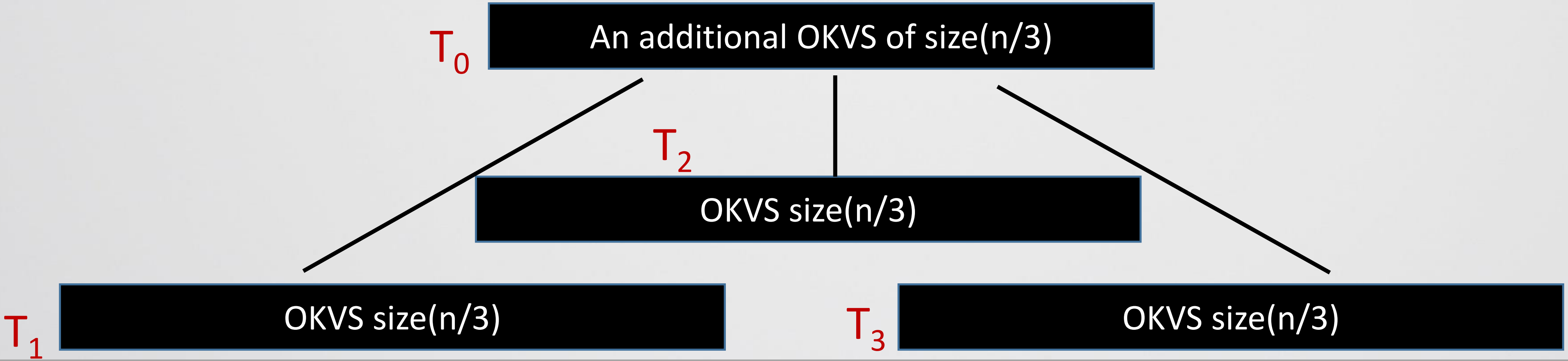
- If encoding succeeds for all of T_1,T_2,T_3, then
  - Fill random values in T_0.
  - Insert values to T_1,T_2,T_3, such that decoding succeeds: for all k, insert to $T_{H(k)}$ the value Decode(T_0,k) XOR value(k)



Star architecture - encoding

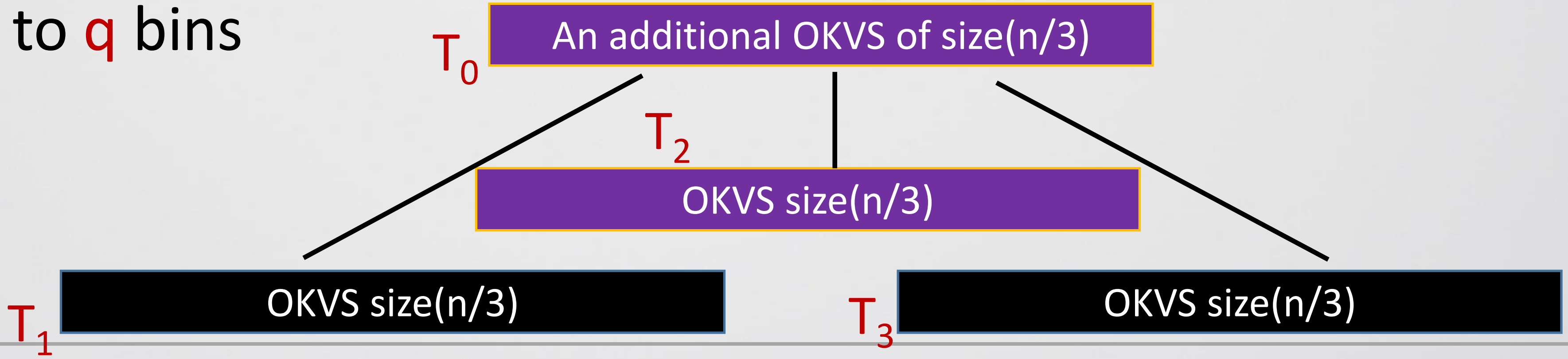
Same if encoding fails for T_1 or T_2

- If encoding succeeds for T_1,T_2 but not for T_3, then
  - Fill random values in T_3.
  - Insert values to T_0, such that decoding succeeds for items mapped to T_3 : for k mapped to T_3, insert to T_0 the value Decode(T_3,k) XOR value(k)
  - Insert values to T_1,T_2, such that decoding succeeds: for all k mapped to T_1,T_2, insert to $T_{H(k)}$ the value Decode(T_0,k) XOR value(k)



Star architecture – bad event

- If encoding fails for two tables, then the process fails
- This only happens with probability $\approx \binom{4}{2} \cdot p^2$
- Performance:
  - Size: 1.33 × optimal OKVS
  - To set the parameters, need to verify a failure probability of p (easier)
  - Obtain smaller failure probability p^2
- Can generalize to q bins

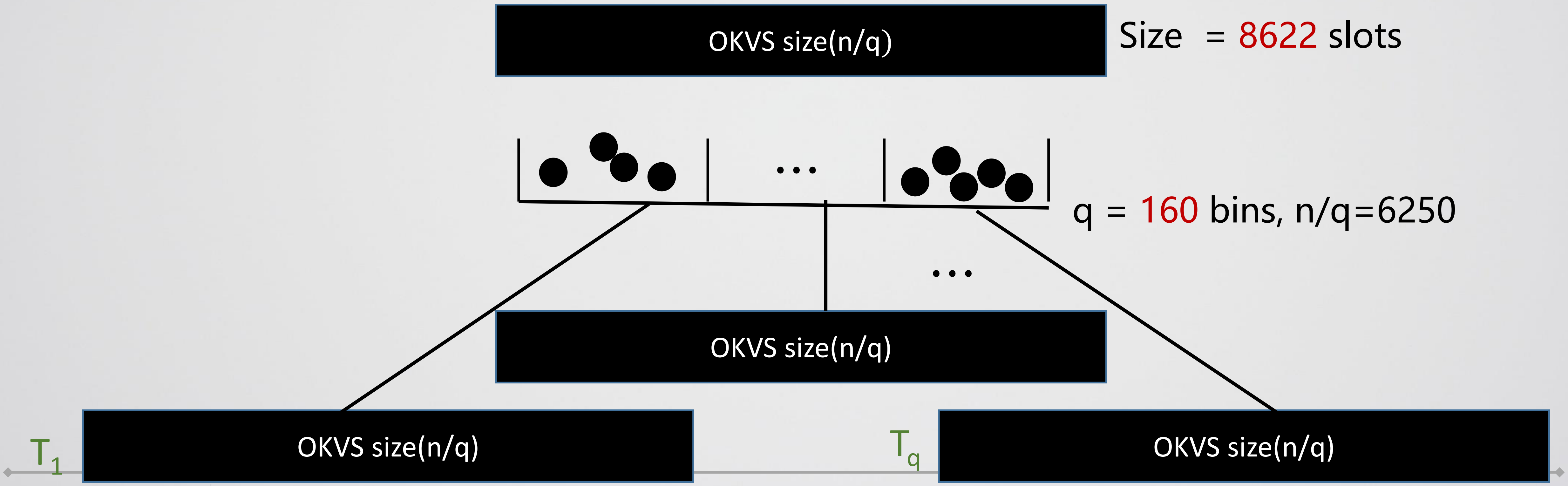


Concrete parameters for OKVS

Encode $n=10^6$ key-value pairs:

Pr[encode fails] $= 2^{-45.05}$ Encoding time = 2.915 s Decoding time = 1.625 seconds

OKVS size(n) = 161x8622 = 1.388n



Further Improvements?

- Can design recursive constructions
- For practical deployments, a single-level construction seems sufficient

- Open question: build a polynomial-size OKVS with a <u>negligible</u> failure probability (polynomial-size amplification of a small OKVS which has a polynomial failure probability?)

Applications of OKVS

Amplified 3H-GCT can replace any random encoding task

Polynomials

- ✓ Sparse OT extension → PSI [PRTY19]
- ✓ Oblivious Programmable PRF
  - ✓ Circuit-PSI [PSTY19, GMRSS21]
  - ✓ Private Set Union [KRTW19]
  - ✓ Multi-party PSI [KMPRT17]

    new efficient malicious-secure n-PSI

PaXoS

- ✓ OT-PaXoS PSI [PRTY20]
  - ✓ fastest semi-honest 2PSI
  - ✓ fastest malicious 2-PSI
  - ✓ empirically verified
  - ✓ generalize to admit linear OKVS

    new vOLE-OKVS PSI
- ✓ vOLE-PaXoS PSI [RS21]

Experimental Results

Takeaways while using this to compute PSI on million items:

3H-GCT, 3H-GCT (star-amp) : 1.61x, 1.43x less communication than PaXoS-PSI

malicious : fastest run-time, ~2x faster than PaXoS-PSI

: faster than [PSTY19] semi-honest PSI

What should we consider when choosing a PSI solution?

Simplicity

- Most cryptographic papers optimize performance, but if you want to use PSI, you would also desire a solution that it is
  - simple to understand and to explain (to your managers)
  - simple to implement
- DH based constructions are much simpler than the constructions based on OT extension + hashing

α

$x_1,\ldots,x_n$

β

$y_1,\ldots,y_n$

$(H(x_1))^\alpha,\ldots,(H(x_n))^\alpha \longrightarrow$

$\longleftarrow (H(y_1))^\beta,\ldots,(H(y_n))^\beta$

$\longleftarrow ((H(x_1))^\alpha)^\beta,\ldots,((H(x_n))^\alpha)^\beta$

$((H(y_1))^\beta)^\alpha,\ldots,((H(y_n))^\beta)^\alpha$

Using probabilistic constructions for PSI

- What is the concrete failure probability?
- Sometimes a heuristic analysis is fine
- For some applications hashing failures do not matter much
  - ML ?
  - CSAM detection (false negatives are fine)
- The failure probability is a function of the input size
  - For small inputs, failure probability might be too large ☹

What input size should we plan for?

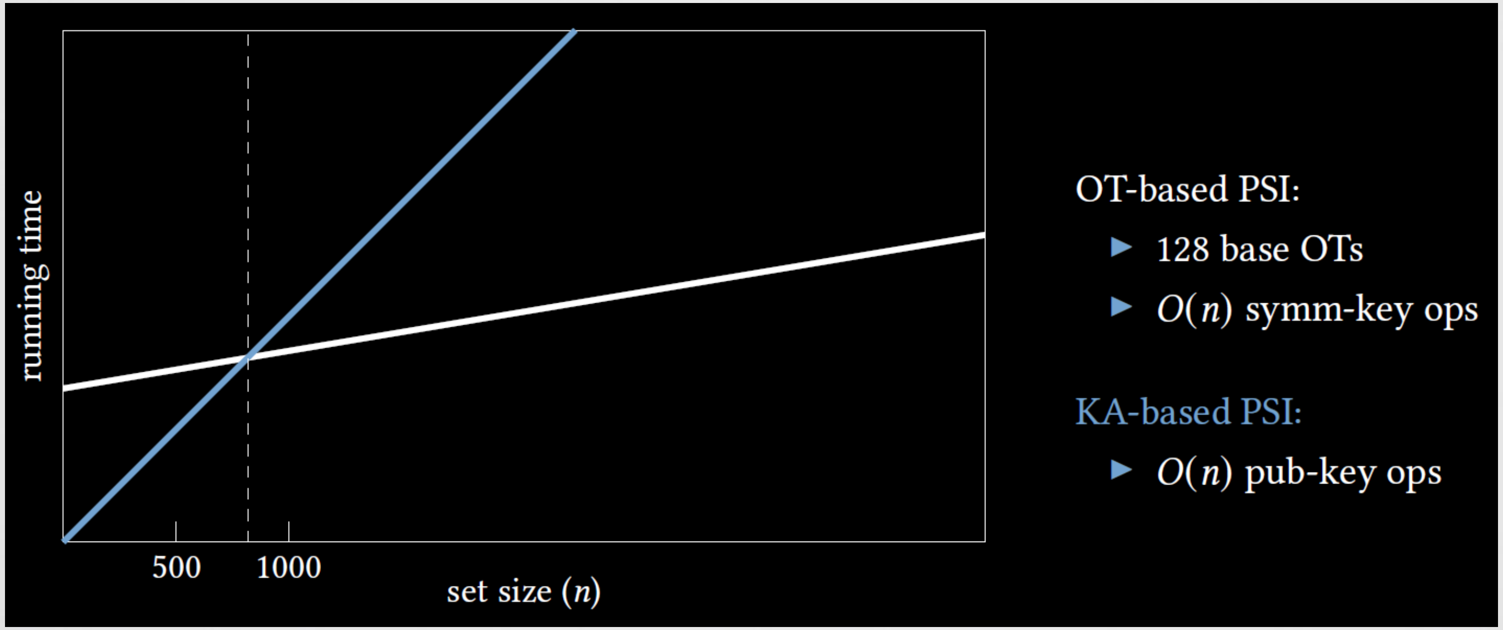
The <u>cost-per-item</u> of PSI for small sets is higher than for larger sets:

- OT extension / VOLE run a preprocessing step using public key operations
  - This is costly if we do only a few hundred OTs
- The hashing failure probability is smaller for larger input sets
  - For smaller sets, obtaining a failure probability of 2^{-40} is costly

→ For smaller input sizes, DH might be better than OT-based PSI

What input size should we plan for?

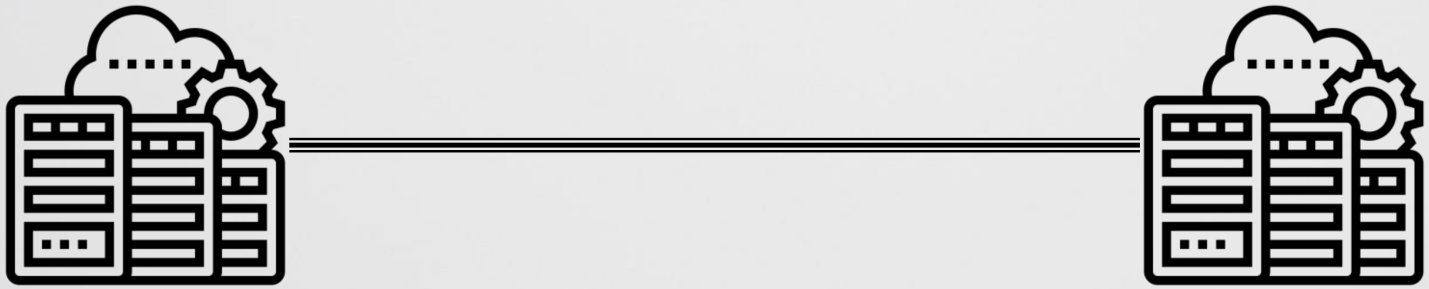
- For smaller input sets, a recent DH-based protocol of Rosulek-Triue (CCS 2021) is best (also has malicious security)



(graph by Mark Rosulek)

How to measure performance?

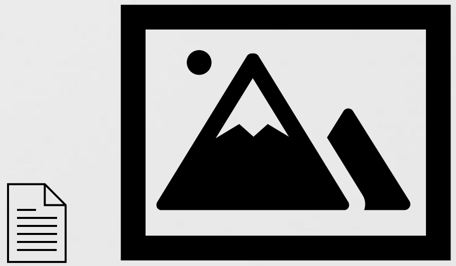
- What is more important, computation or communication costs?
- Google [IKN+19]: "For the offline "batch computing" scenarios we consider, **communication costs are far more important** than computation. … It is much less expensive to add CPUs to a shared network than to expand network capacity."



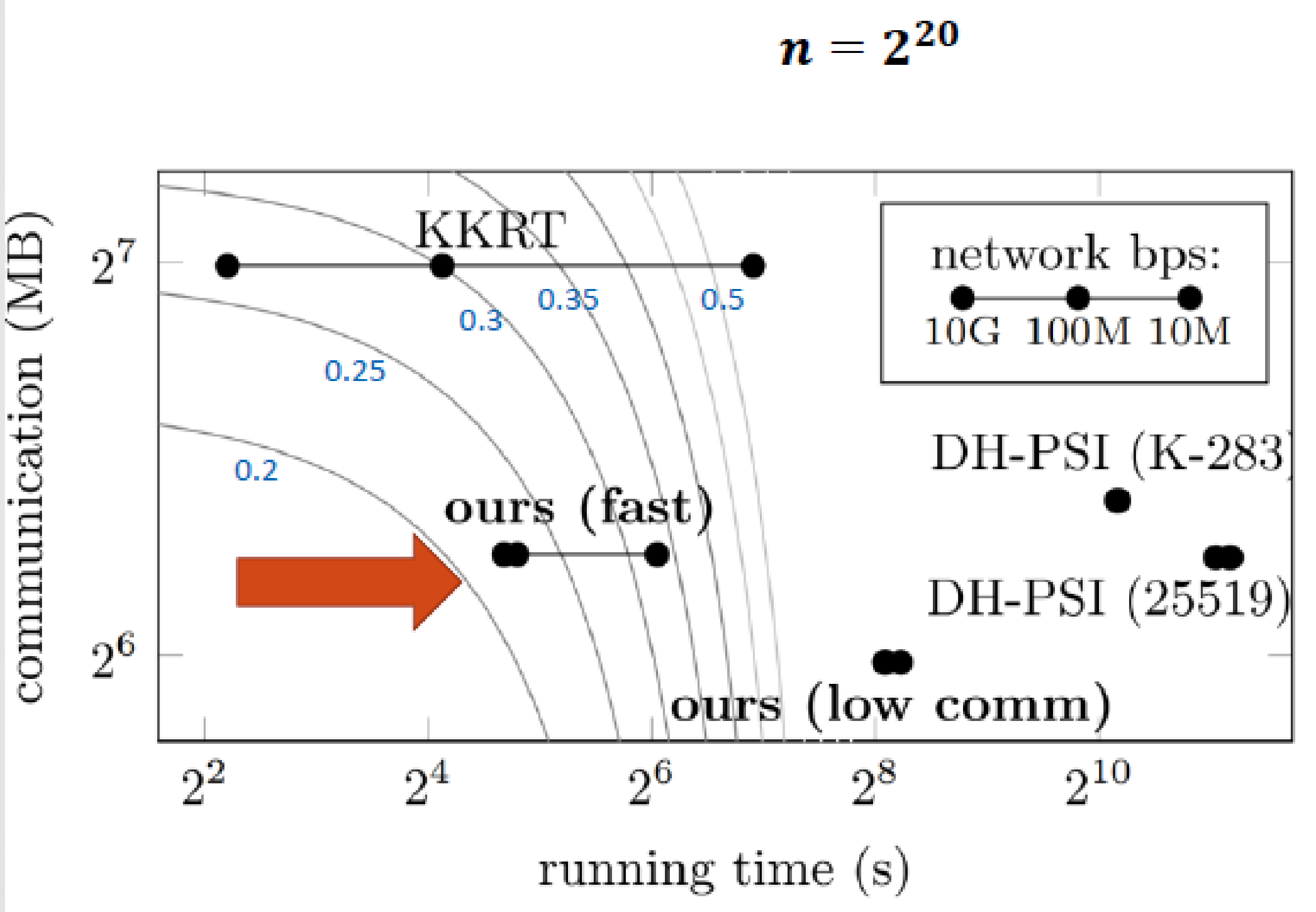
How to measure performance?

Apple's recent CSAM detection system:

- Each photo uploaded from a device is accompanied by a PSI message
- The additional message size is negligible. Computation (=battery usage) is far more important.



SpOT PSI (Crypto 2019 [PRTY])



Security: Semi-honest vs. Malicious

- For PSI, the performance gap between semi-honest and malicious security is very small ☺

- OT-based protocols: [PRTY20,GPRTY21] have the best performance, and almost the same overhead for malicious and semi-honest security

- DH-based protocols: for small sets, the malicious protocol of [RT21] is only 10%-20% slower than the best semi-honest PSI protocol

What should we use?

DH-based protocols

- Best performance for small inputs
- Easy to implement and explain
- Can be modified to compute intersection size

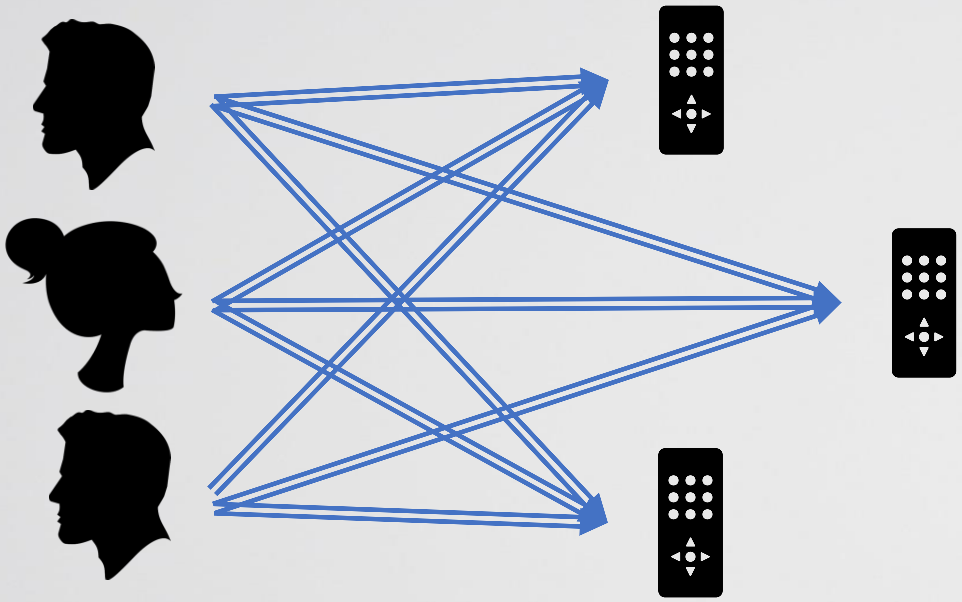
OT-based protocols

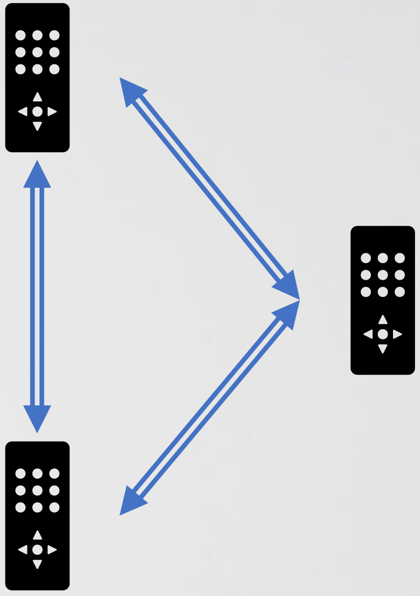
- Much more efficient for larger inputs
- More complicated
- Harder to modify

PSI + generic MPC protocols

- Can compute arbitrary functions
- Slower than OT-based
- More complicated

A different model: Outsourced PSI





- “MPC as a service”
- Many users share their data between servers, which a run the MPC.
- A different trust assumption (!) but can be very efficient !