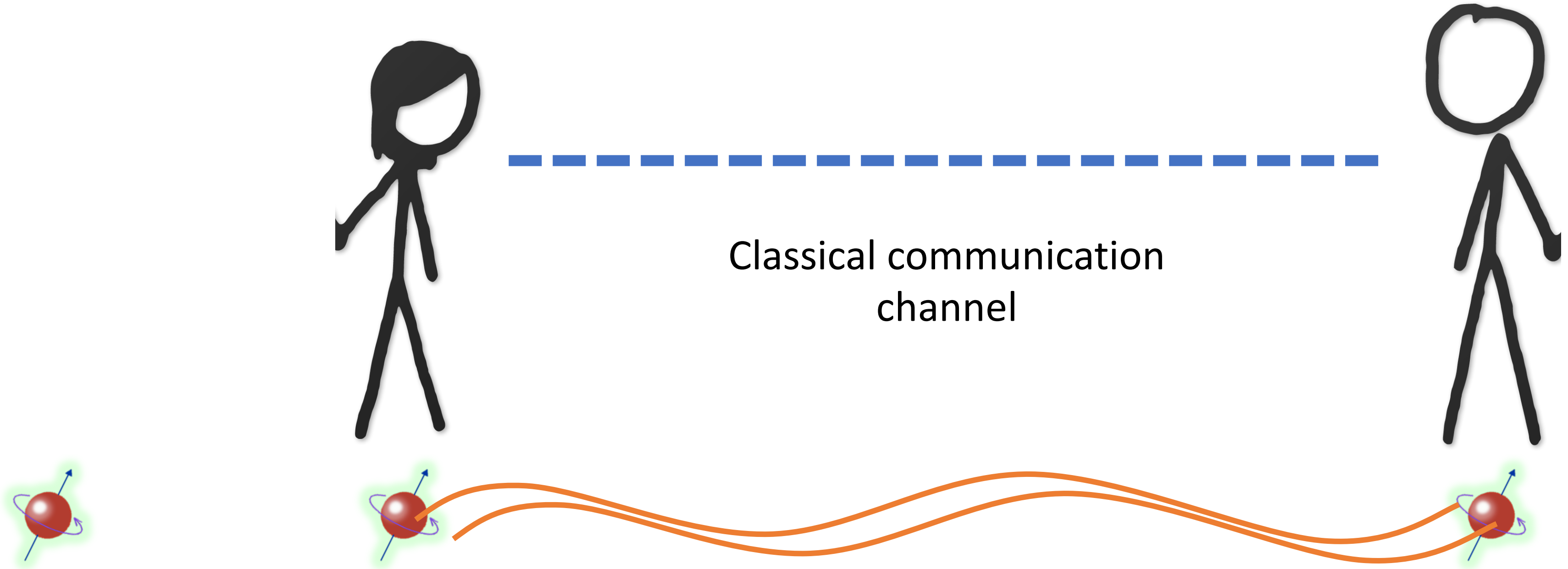
Crash Course in Quantum Computing

Hour 2: Quantum Computation

BIU Winter School on Cryptography 2021

Lecturer: Henry Yuen

Quantum Teleportation



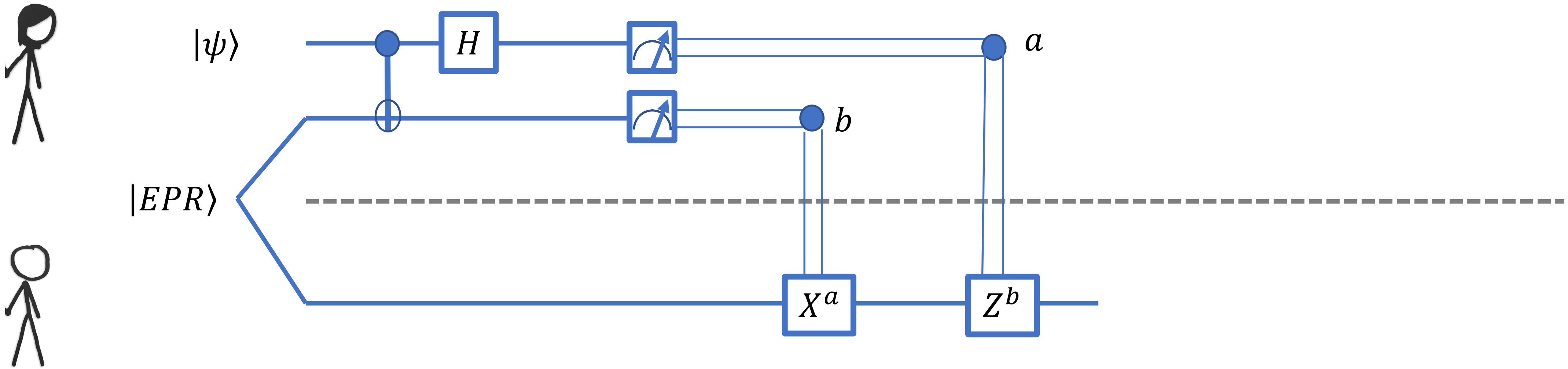
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Shared before protocol begins.

Quantum teleportation allows Alice to send $|\psi\rangle$ to Bob using preshared entanglement and classical communication.

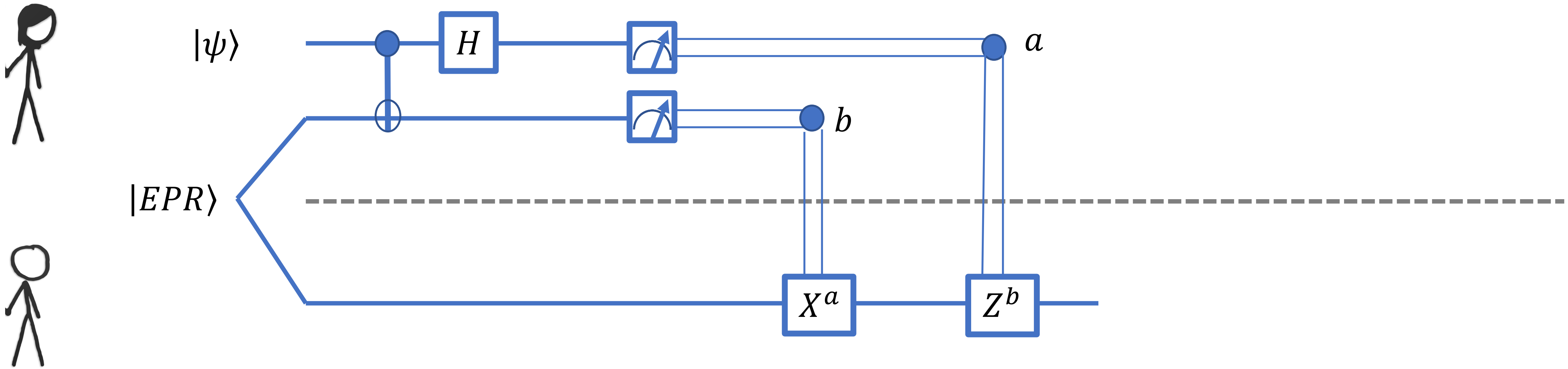
Quantum Teleportation



Quantum circuits:

- Each horizontal wire represents a qubit
- Time runs from left to right
- Initial state of qubits is written on left hand side

Quantum Teleportation



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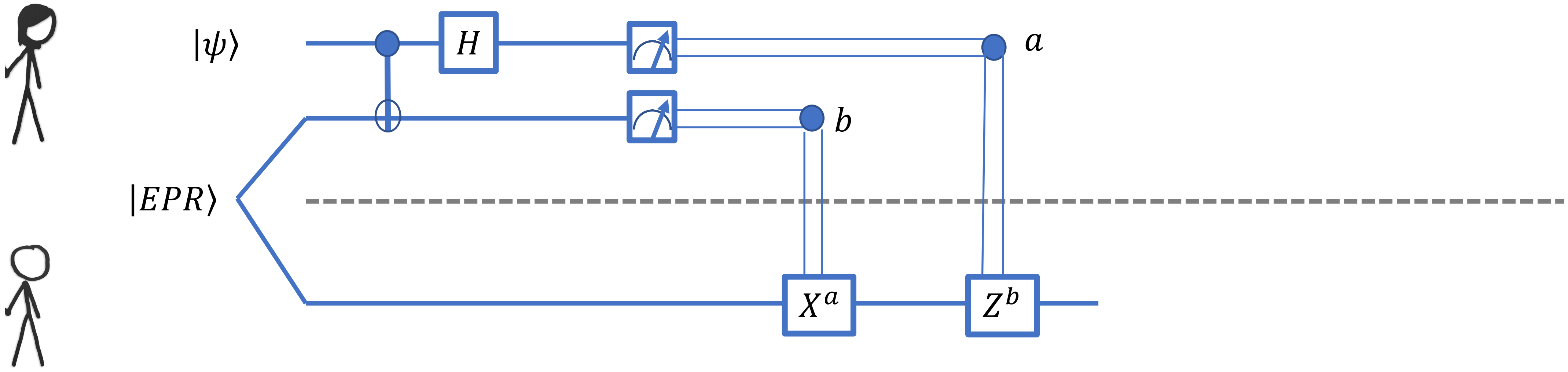
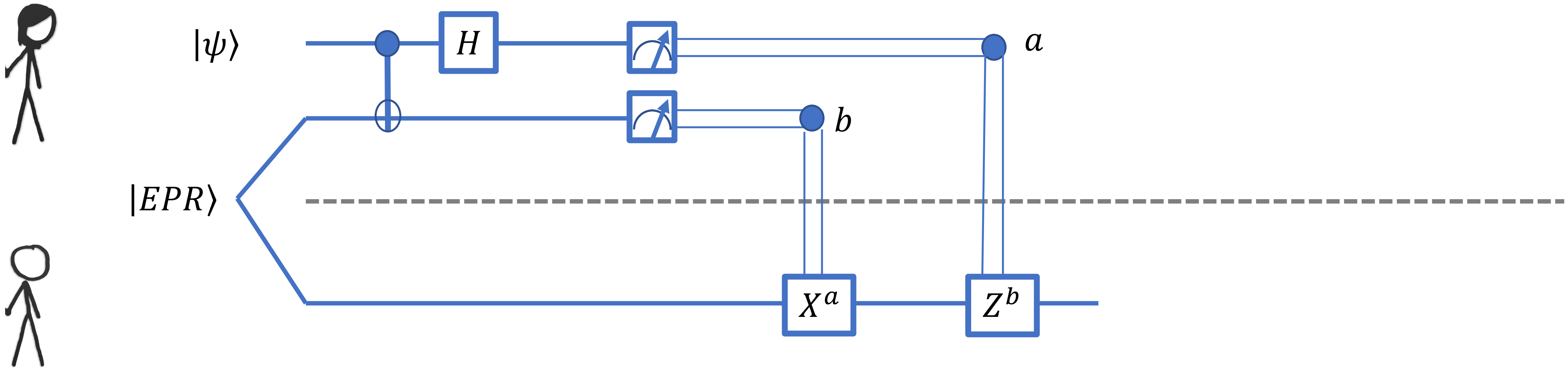


Claim: At the end of protocol, Bob has $|\psi\rangle$.

Quantum Teleportation



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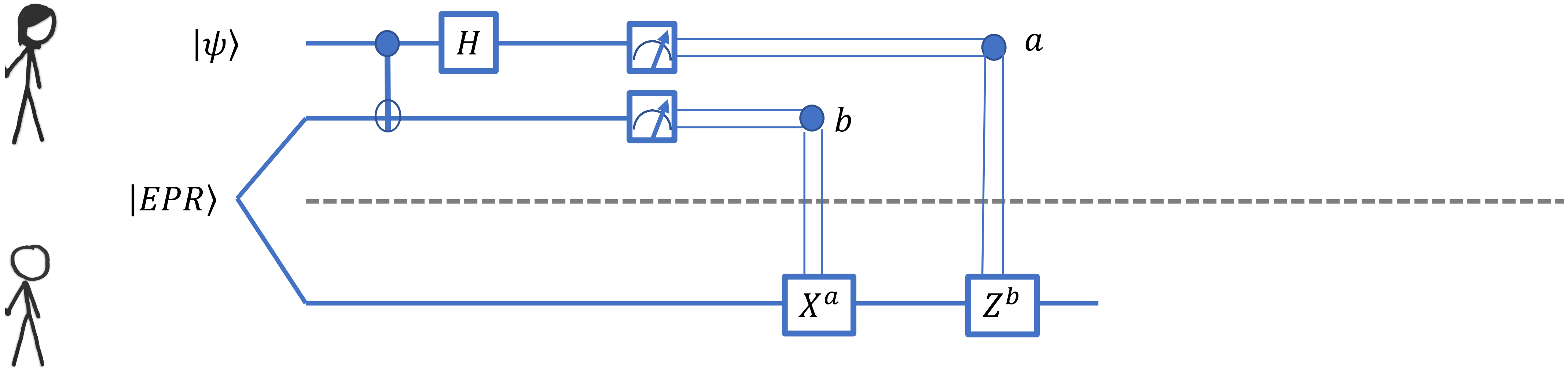

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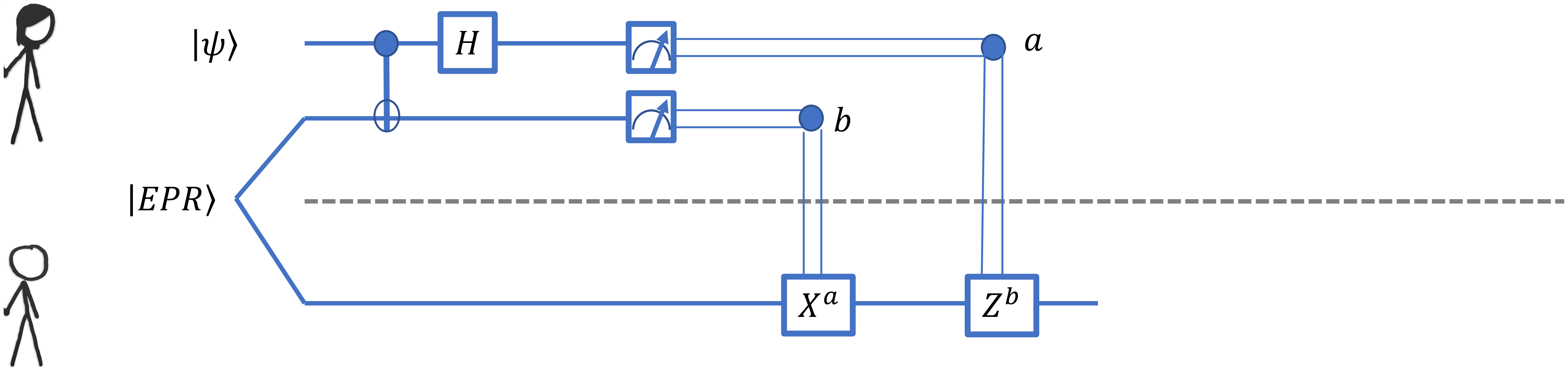

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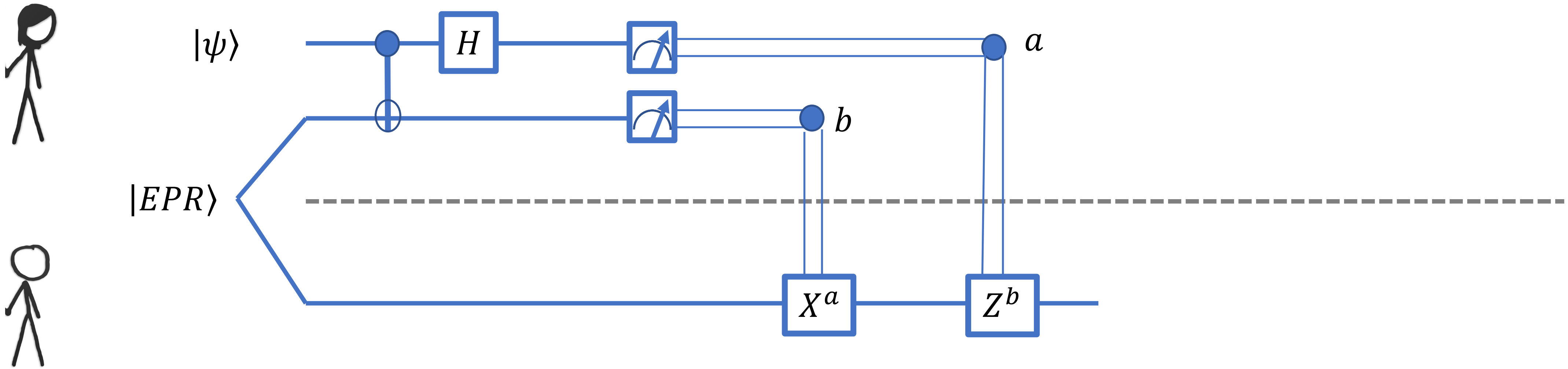


Claim: At the end of protocol, Bob has $|\psi\rangle$.

Quantum teleportation does not allow Alice to instantaneously send $|\psi\rangle$ to Bob.
Alice needs to communicate classical bits to Bob!

Quantum Circuit Model

Quantum gates

- A k qubit-*quantum gate* is a $2^k \times 2^k$ unitary matrix U

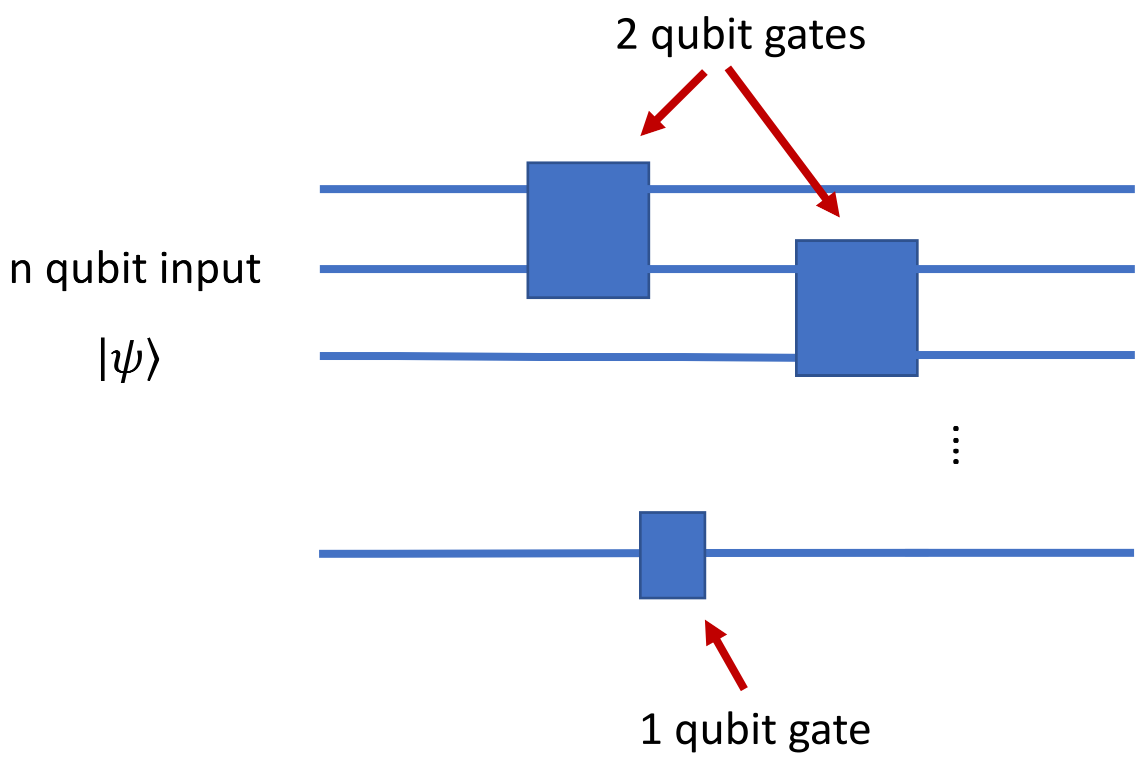
- Common single-qubit quantum gates:
  - I – identity
  - X – bitflip: $|0\rangle \leftrightarrow |1\rangle$
  - H – Hadamard: $|0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
    $|1\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Phase gates

$$Z: |0\rangle \mapsto |0\rangle, \quad |1\rangle \mapsto -|1\rangle$$
$$P: |0\rangle \mapsto |0\rangle, \quad |1\rangle \mapsto i|1\rangle$$
$$T: |0\rangle \mapsto |0\rangle, \quad |1\rangle \mapsto e^{\frac{2\pi i}{8}}|1\rangle$$

- Two-qubit gates:
  - $CNOT$ – controlled NOT operation: $CNOT|x, a\rangle = |x, a \oplus x\rangle$

Quantum circuits

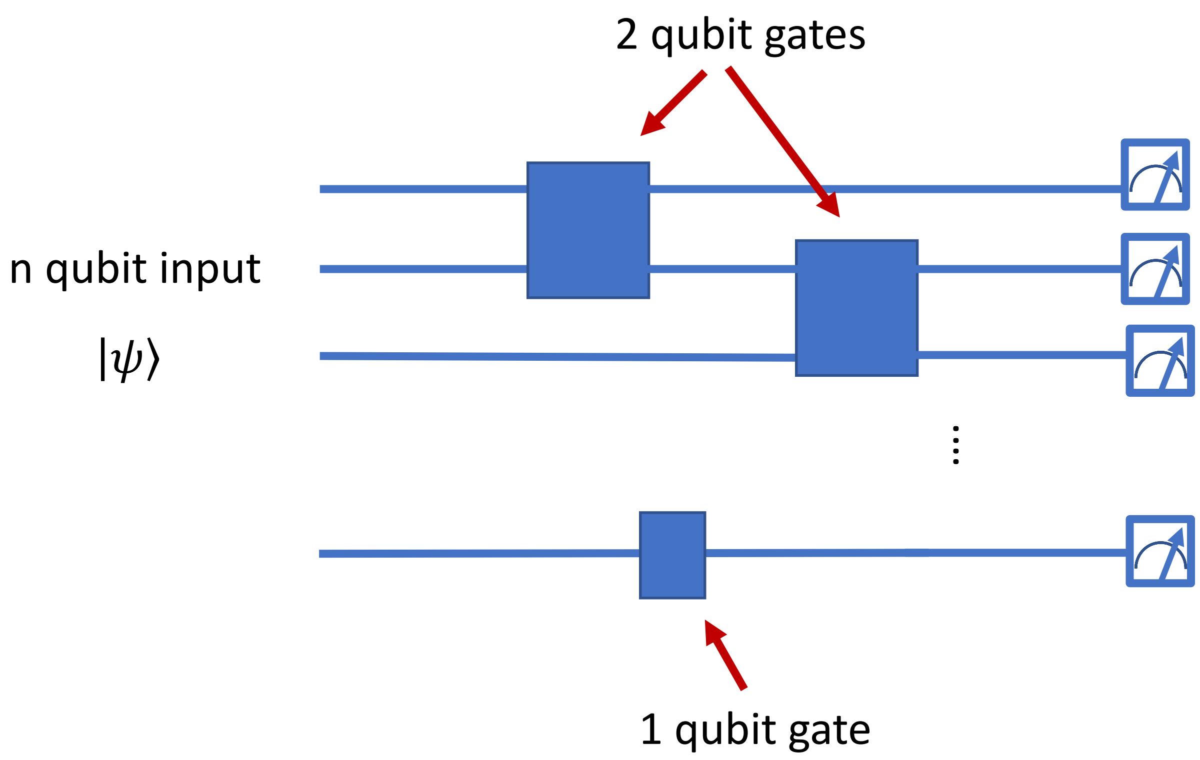


- A quantum circuit F consists of an ordered collection of 1- and 2-qubit gates $G_1, G_2, \ldots$ applied to subsets of qubits.

- Output of circuit F on input $|\psi\rangle$ is equal to

$$G_m \cdots G_2 G_1 |\psi\rangle$$

Measurements



- At end of computation, if final state is

$$|\varphi\rangle = \sum \beta_x \, |x\rangle$$

can perform **measurement** to get classical outcome of computation.

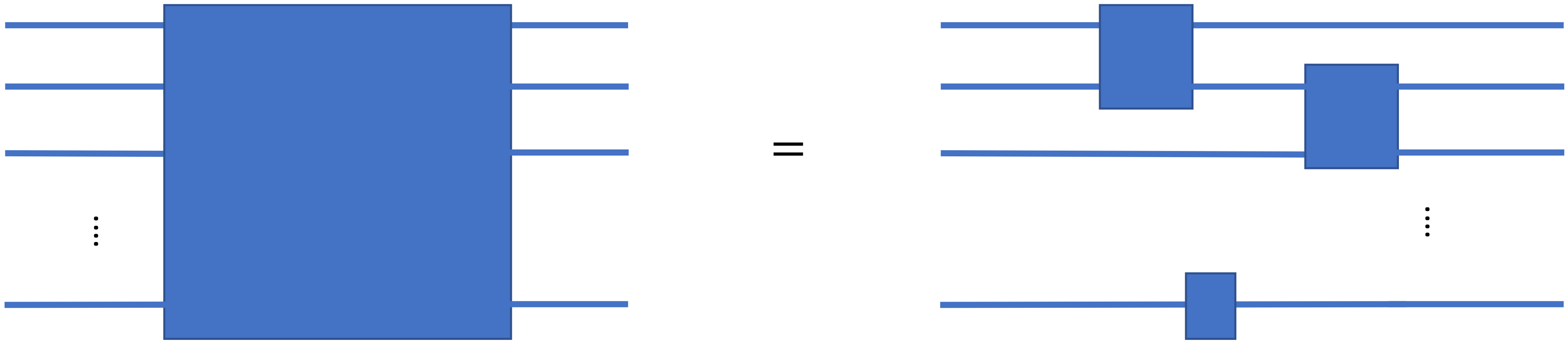
- Measurement is probabilistic: obtains outcome $x \in \{0,1\}^n$ with probability $|\beta_x|^2$.

- Measurement is **destructive**: measuring in middle of quantum computation will disturb the state.

We can also allow intermediate measurements (like in quantum teleportation), but for now let's assume that measurements happen at the very end.

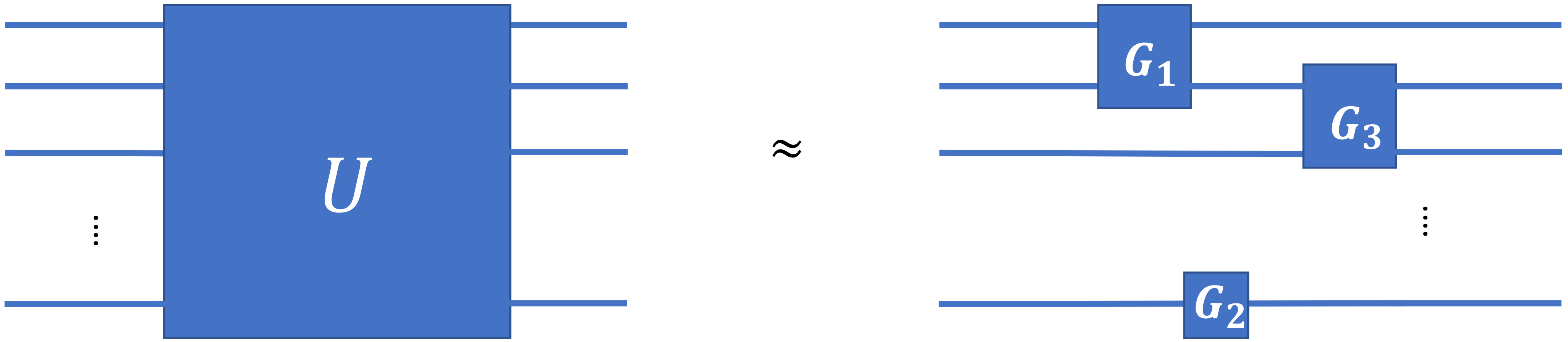
Universal and non-universal gate sets

- Every n-qubit unitary U can be implemented as a quantum circuit consisting of single-qubit gates and CNOT.
- In worst case, such a circuit requires $\approx 4^n$ gates.
- Can use arbitrary single-qubit gates $G \in \mathbb{C}^{2\times 2}$.



Universal and non-universal gate sets

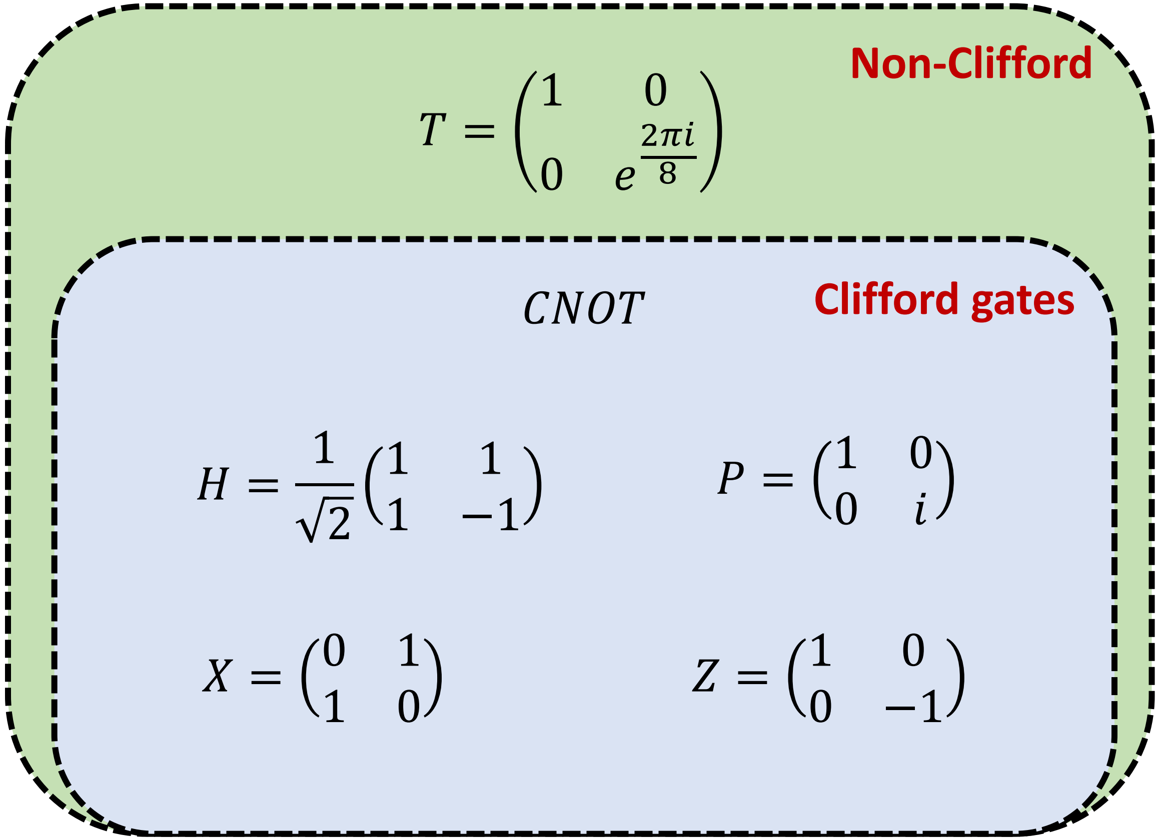
- In practice, we can only use gates from a fixed, finite set (depending on your hardware).
- A set Λ of gates is ***universal*** if any unitary (on any number of qubits) can be approximated arbitrarily well by a circuit consisting of gates from Λ.
- A unitary U ϵ-approximates another unitary V if: $\max_{|\psi\rangle} \| U|\psi\rangle - V|\psi\rangle \| \leq \epsilon$



$G_1, G_2, G_3, \ldots \in \Lambda$

Universal and non-universal gate sets

- **Ex**: $\Lambda = \text{Clifford} \cup \{T\}$ is a universal gate set!
  - Clifford= gates generated by $\{H, P, CNOT\}$



Universal and non-universal gate sets

- **Ex**: $\Lambda =$ Clifford is *not* universal gate set.
  - Clifford= gates generated by $\{H, P, CNOT\}$

$CNOT$ **Clifford gates**

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

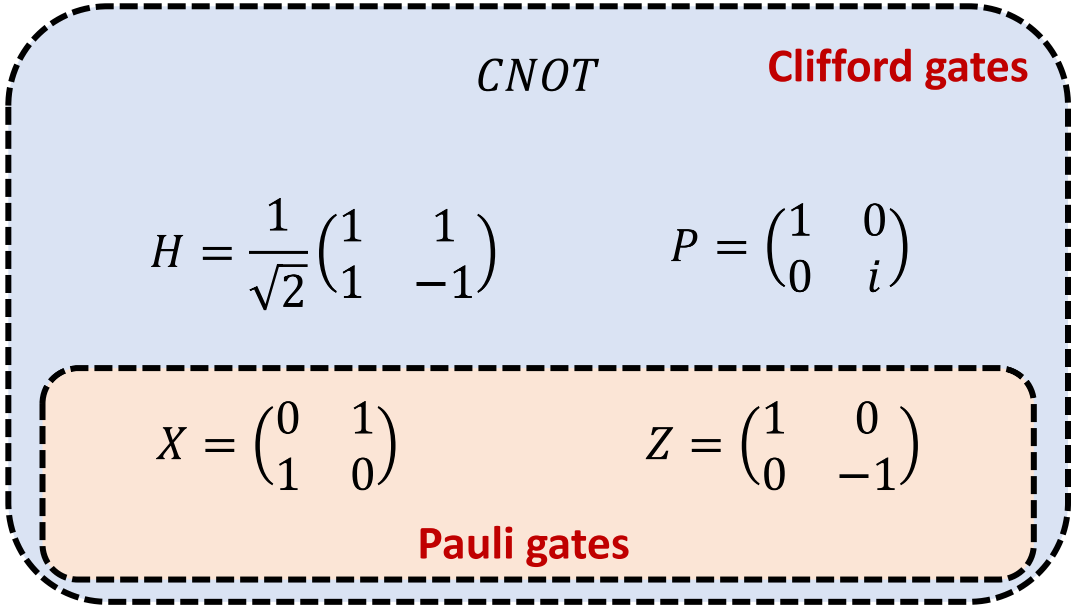
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Fact #0: Clifford circuits are not even universal for *classical* computation.

Fact #1: Clifford circuits (with all zeroes input) can be efficiently simulated on classical computers (Gottesman-Knill Theorem).

Universal and non-universal gate sets

- **Ex**: $\Lambda =$ Clifford is *not* universal gate set.
  - Clifford$=$ gates generated by $\{H, P, CNOT\}$
- Pauli $=$ gates generated by $\{X, Z\} \subseteq$ Clifford
- n-qubit Pauli unitaries: tensor products of $\{I, X, Y, Z\}$



Fact #2: Clifford circuits/unitaries are equivalently defined in terms of their behavior on ***Pauli matrices***.

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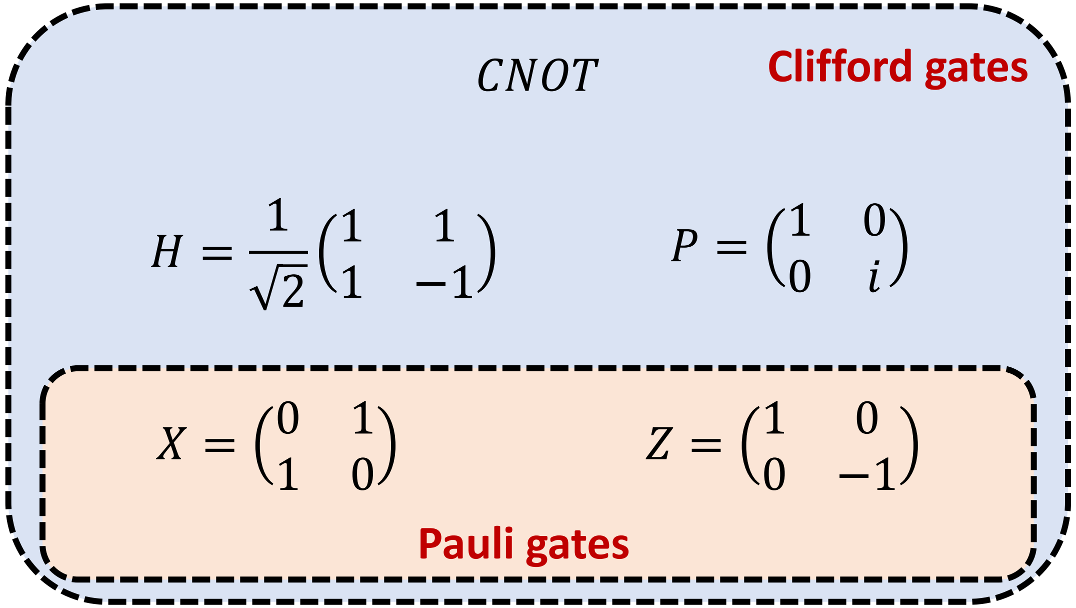

Fact #2: Clifford circuits/unitaries are equivalently defined in terms of their behavior on ***Pauli matrices***.

For all Pauli unitaries $W = W_1 \otimes W_2 \otimes \cdots \otimes W_n$

for all n-qubit Clifford unitaries C, there exists another Pauli unitary $W' = W_1{}' \otimes W_2{}' \otimes \cdots \otimes W_n{}'$ such that

$$WC = CW'$$

Ex:

Computing classical functions, quantumly

How to compute $f: \{0,1\}^n \to \{0,1\}^m$ using a quantum circuit?

Can call classical functions as a subroutine using ***classical oracles***: define the unitary U_f on $n + m$ qubits: for all $x \in \{0,1\}^n, b \in \{0,1\}^m$,

$$U_f|x, b\rangle = |x, b \oplus f(x)\rangle$$

Ex: $f = \mathrm{AND}, f = \mathrm{NOT}$

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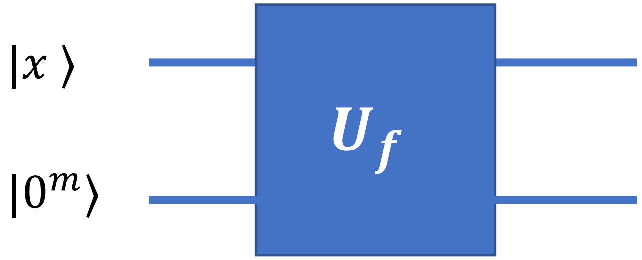

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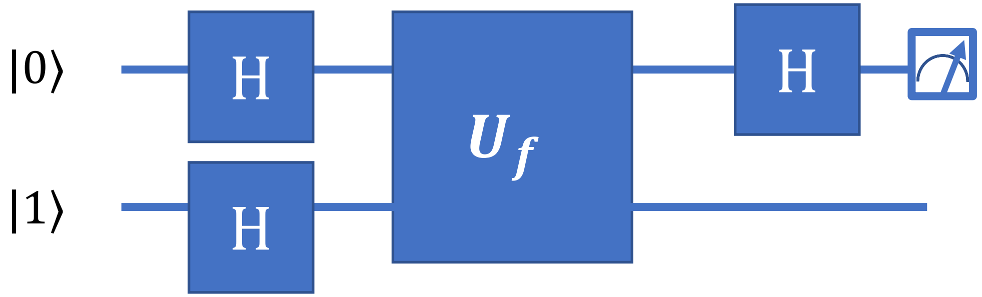
Size s classical circuit computing $f \implies$ There is a size $O(s)$ quantum circuit computing U_f.

Computing classical functions, quantumly

Magic starts happening when classical oracles are queried on a ***superposition*** of inputs.

Deutsch's Problem: Given $f: \{0,1\} \rightarrow \{0,1\}$, determine using one quantum query to U_f whether

- <u>YES case:</u> $f(0) \neq f(1)$
- <u>NO case:</u> $f(0) = f(1)$



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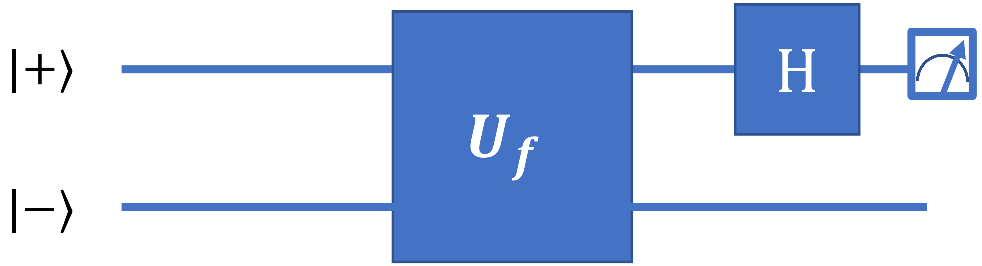
circuit equivalent to



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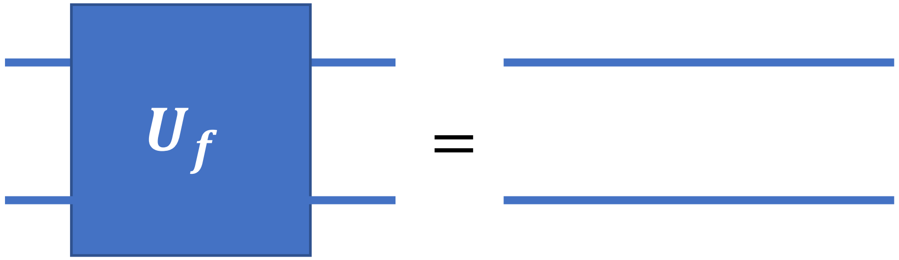
Case 1: $f(0) = f(1) = 0$



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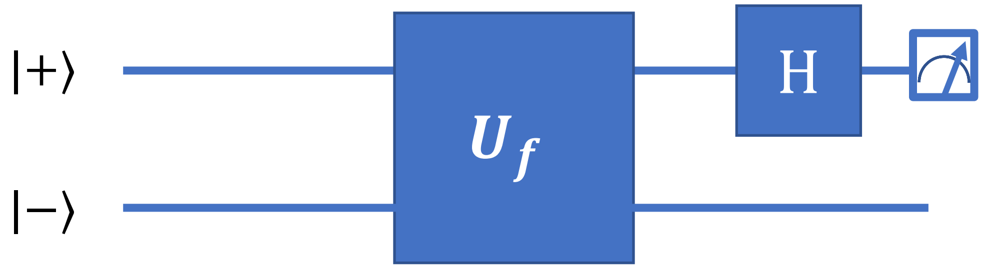

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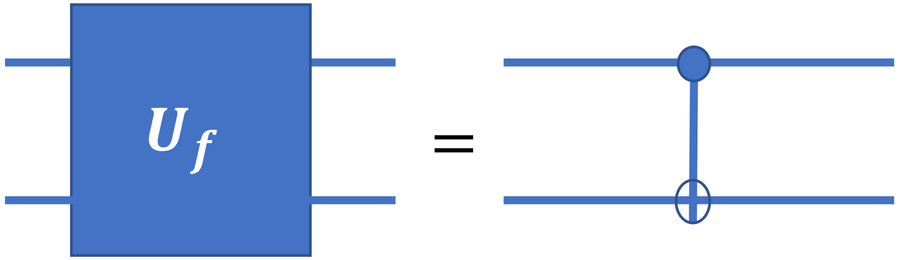
Case 2: $f(0) = 0, f(1) = 1$



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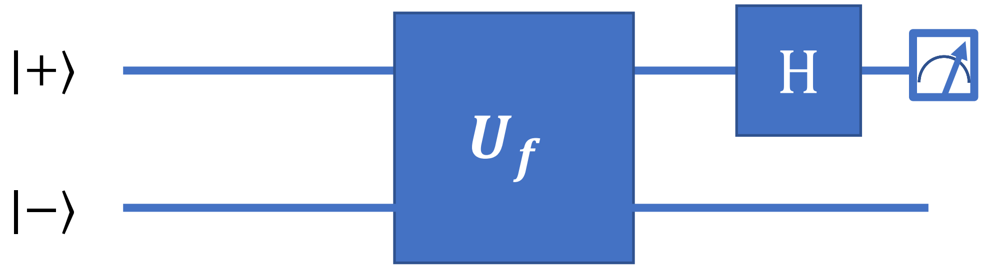

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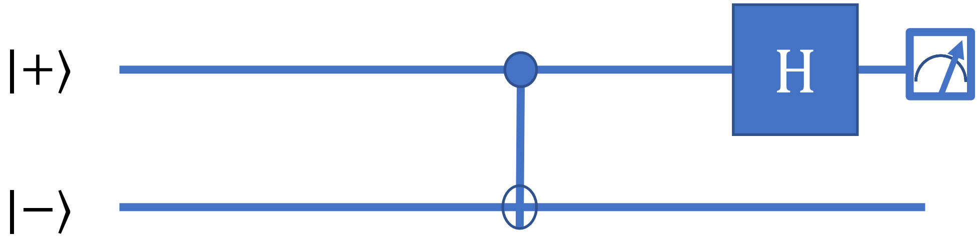


Grover Search

Unstructured search

Search problem: Given black-box access to $f: \{0,1\}^n \to \{0,1\}$, find x such that $f(x) = 1$.

Classical query complexity: $\Omega(2^n)$

Quantum query complexity: $O(\sqrt{2^n})$

Unstructured search

Search problem: Given black-box access to $f: \{0,1\}^n \rightarrow \{0,1\}$, find x such that $f(x) = 1$.

For boolean functions, we can use different oracle (called *phase* oracle): for all $x \in \{0,1\}^n$

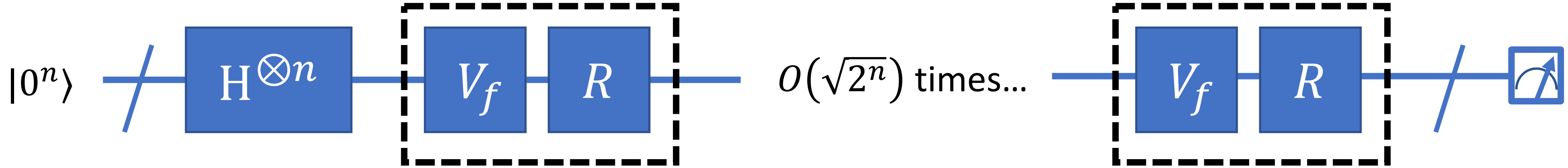
$$V_f|x\rangle = (-1)^{f(x)}|x\rangle$$

XOR oracles and phase oracles are equivalent!

Unstructured search

Search problem: Given black-box access to $f: \{0,1\}^n \to \{0,1\}$, find x such that $f(x) = 1$.

Assume there exists a unique x^* such that $f(x^*) = 1$.



Grover iterate

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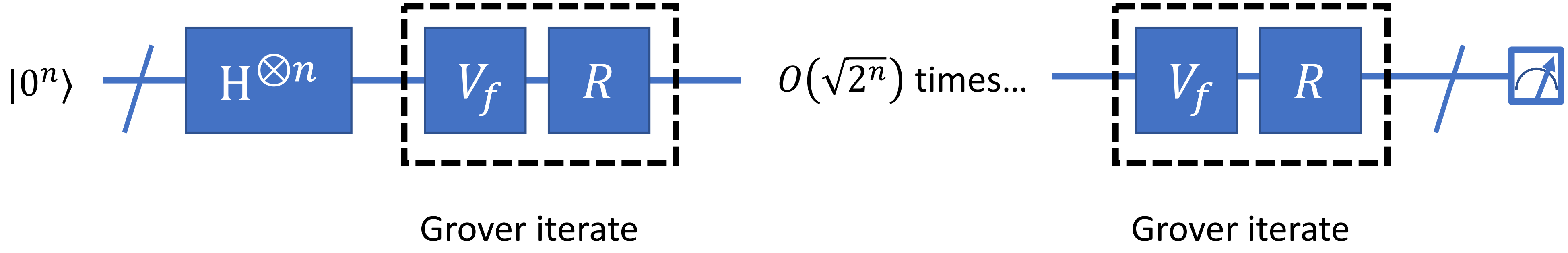

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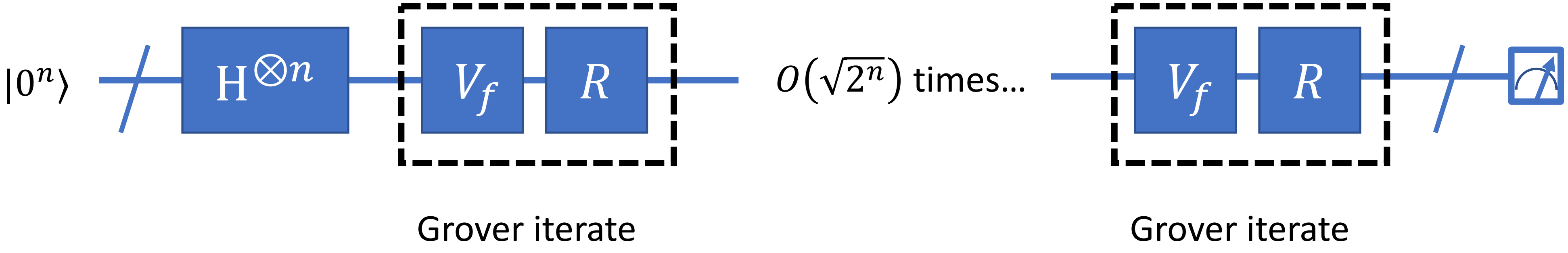
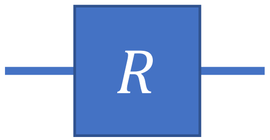


”diffusion operator”, “inversion about the mean”,…



$= 2|s\rangle\langle s| - I$

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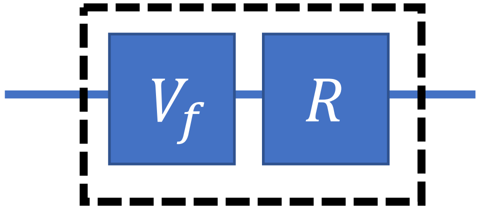
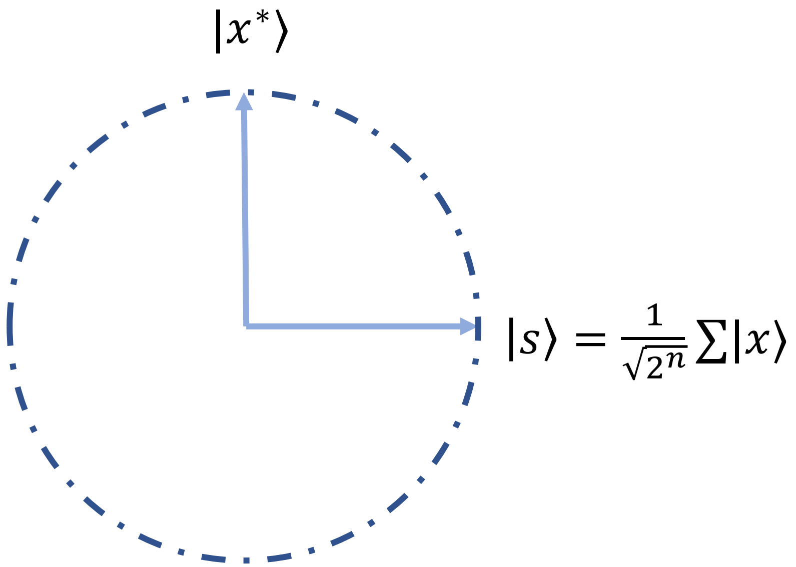
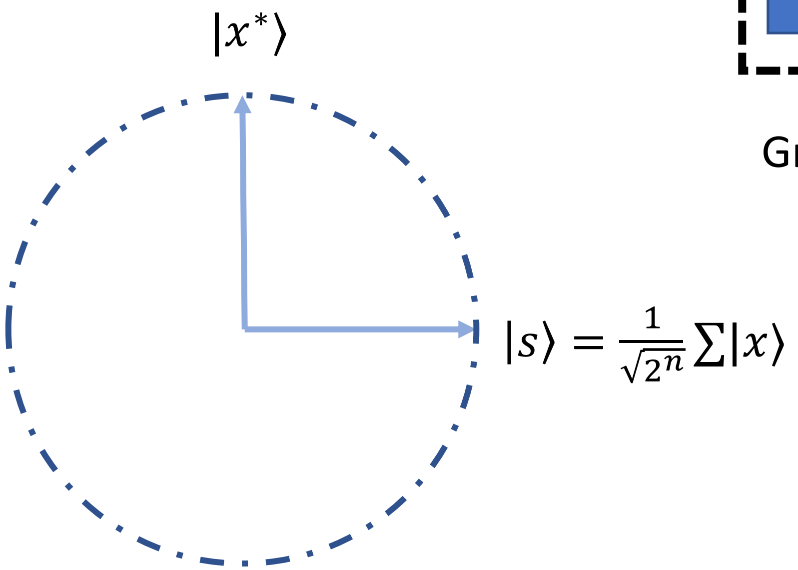


Grover iterate



Starting state of algorithm



After one Grover iterate

Unstructured search

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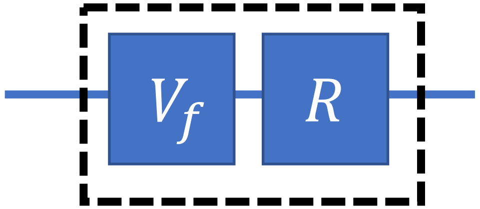

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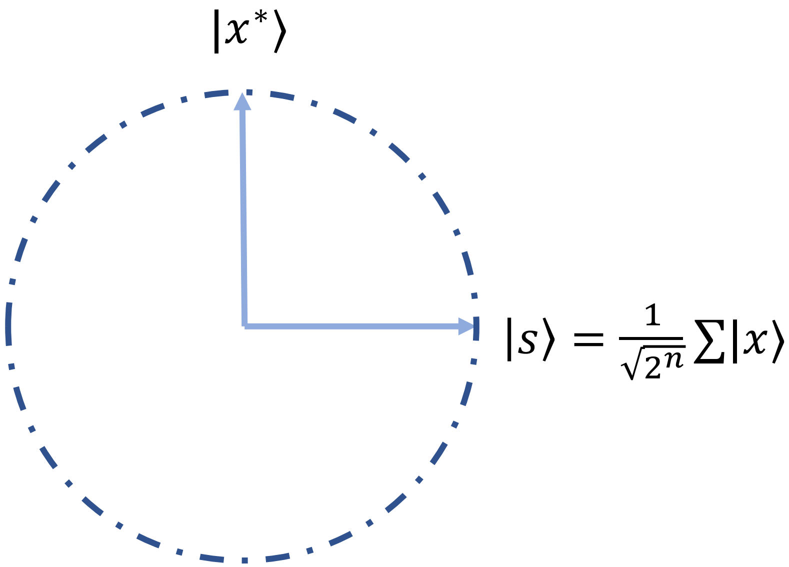

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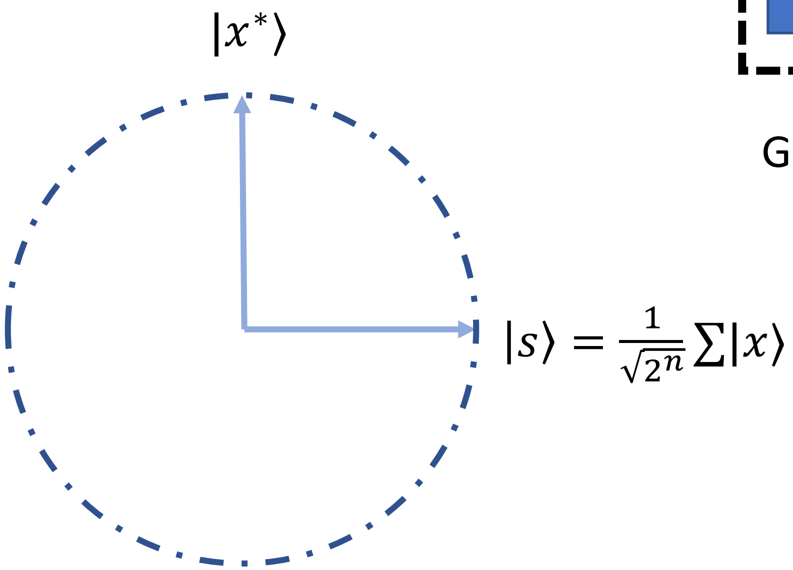

after $O\left(\sqrt{2^n}\right)$ iterates

Unstructured search

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What if there are T solutions x such that $f(x) = 1$?

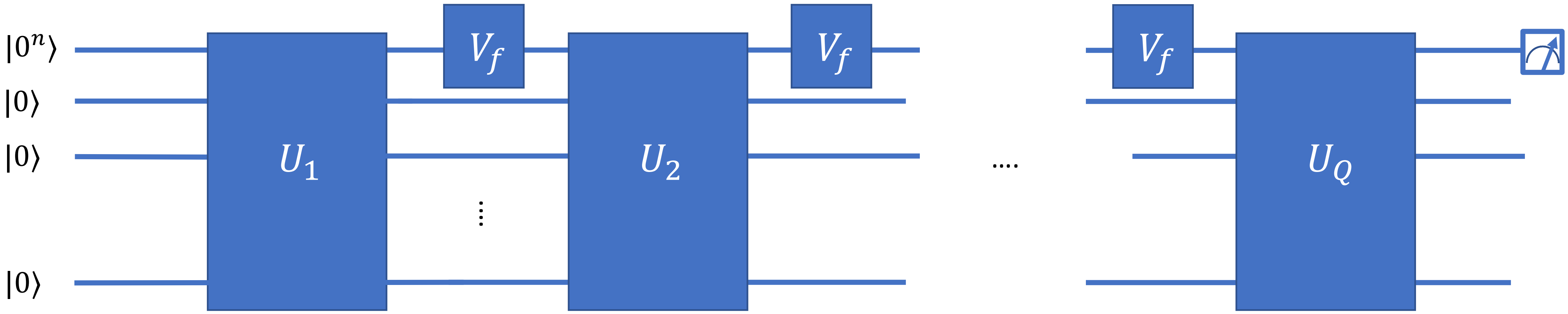
- If T is known before hand, run $O\left(\sqrt{\frac{2^n}{T}}\right)$ queries.

- If T is unknown, then using more clever algorithm, can still find solution in $O\left(\sqrt{\frac{2^n}{T}}\right)$ queries!

Quantum lower bound for unstructured search

Grover's algorithm is optimal (in terms of query complexity) for solving the unstructured search problem: $\Omega\left(\sqrt{2^n}\right)$ queries are needed!

Bennett, Brassard, Bernstein, Vazirani proved this using a *hybrid argument*. Suppose there was a Q-query algorithm for unstructured search, for $Q \ll \sqrt{2^n}$.



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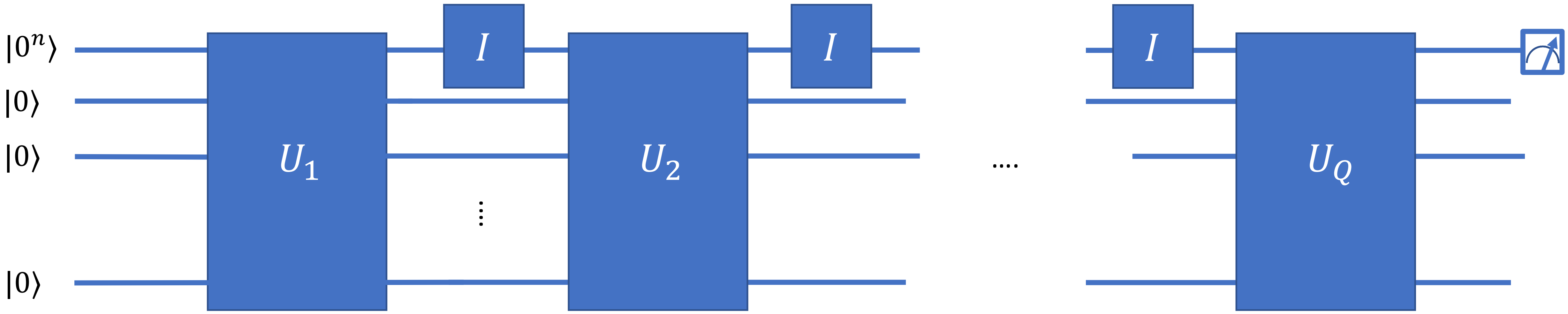

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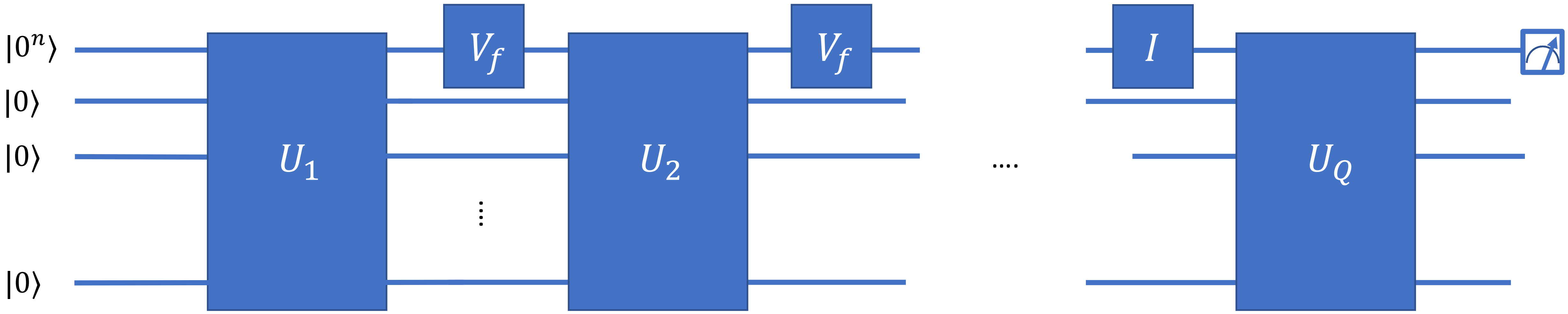

Generalizations of Grover search

- Quantum Counting

- Amplitude Amplification