Session #6: Another Application of LWE: Pseudorandom Functions

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Bar-Ilan University, Israel
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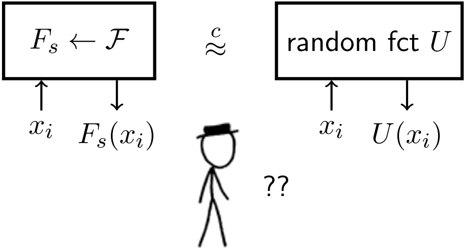
Pseudorandom Functions [GGM'84]

- A family $\mathcal{F} = \{F_s : \{0,1\}^k \to D\}$ s.t. given adaptive query access,



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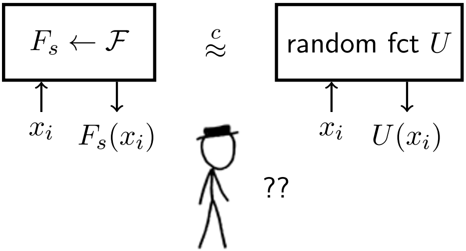

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- Countless applications in symmetric cryptography: (efficient) encryption, authentication, friend-or-foe . . .

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   - ✗ Huge circuits that need much preprocessing
   - ✗ No "post-quantum" construction under standard assumptions

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- Lattices $\Rightarrow$ simple, highly parallel, practically efficient ... PRFs?

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2. Main technique: "derandomization" of LWE: deterministic errors

Synthesizers and PRFs [NaorReingold'95]

Synthesizer

- A deterministic function $S\colon D \times D \to D$ s.t. for any $m =$ poly: for uniform $a_1, \ldots, a_m,\ b_1, \ldots, b_m \leftarrow D$,

$$\{\, S(a_i\,,\, b_j)\,\} \overset{c}{\approx} \text{Unif}(D^{m \times m}).$$

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vs.

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- Alternative view: an (almost) length-squaring PRG with locality: maps $D^{2m} \to D^{m^2}$, and each output depends on only 2 inputs.

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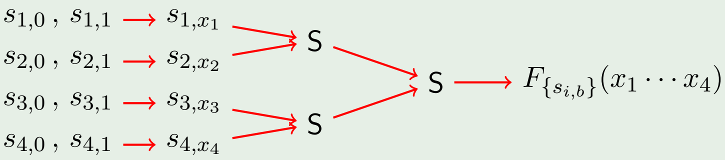

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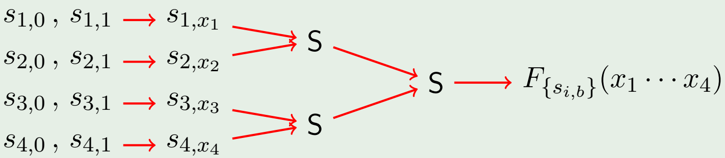

- Security: the queries $F_\ell(x_\ell)$ and $F_r(x_r)$ define (pseudo)random inputs $a_1, a_2, \ldots \in D$ and $b_1, b_2, \ldots \in D$ to synthesizer S.

LWE $\Rightarrow$ Synthesizer?

- Hard to distinguish pairs $(\mathbf{a}_i \in \mathbb{Z}_q^n \, , \, b_i = \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i)$ from $(\mathbf{a}_i \, , \, b_i)$.

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✗ What about $\mathbf{E}_{i,j}$?

Synthesizer must be deterministic. . .

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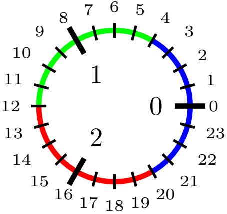

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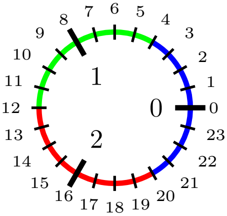

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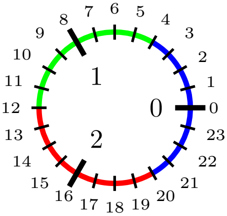

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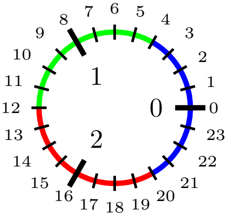

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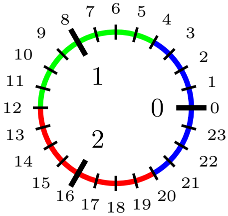

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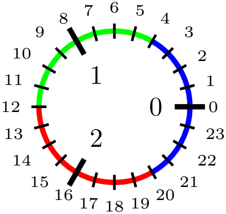

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  Proof idea: w.h.p., $(\mathbf{a}, \lfloor \langle \mathbf{a}, \mathbf{s} \rangle + e \rceil_p) = (\mathbf{a}, \lfloor \langle \mathbf{a}, \mathbf{s} \rangle \rceil_p)$

  and $(\mathbf{a}, \lfloor \text{Unif}(\mathbb{Z}_q) \rceil_p) = (\mathbf{a}, \text{Unif}(\mathbb{Z}_p))$

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- "Tower" of public moduli $q_d > q_{d-1} > \cdots > q_0$.
- Secret key is $2k$ square matrices $\mathbf{S}_{i,b}$ over $\mathbb{Z}_{q_d}$ for $i \in [k]$, $b \in \{0,1\}$.

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- Depth $d = \lg k$ tree of LWR synthesizers:

$$F_{\{\mathbf{S}_{i,b}\}}(x_1 \cdots x_8) =$$
$$\left\lfloor \left\lfloor \lfloor \mathbf{S}_{1,x_1} \cdot \mathbf{S}_{2,x_2} \rceil_{q_2} \cdot \lfloor \mathbf{S}_{3,x_3} \cdot \mathbf{S}_{4,x_4} \rceil_{q_2} \right\rceil_{q_1} \cdot \left\lfloor \lfloor \mathbf{S}_{5,x_5} \cdot \mathbf{S}_{6,x_6} \rceil_{q_2} \cdot \lfloor \mathbf{S}_{7,x_7} \cdot \mathbf{S}_{8,x_8} \rceil_{q_2} \right\rceil_{q_1} \right\rceil_{q_0}$$

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- [NR'97,NRR'00]: direct PRFs from DDH / factoring, in $\mathsf{TC}^0 \subseteq \mathsf{NC}^1$.

$$F_{g,s_1,\ldots,s_k}(x_1 \cdots x_k) = g^{\prod s_i^{x_i}}$$

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Ring variant has small(ish) TC^0 circuit, practical implementation

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  $\Rightarrow$ New function $F'(x) = \lfloor \mathbf{A}_{x_1}\mathbf{S}_2^{x_2} \cdots \mathbf{S}_k^{x_k} \rceil_p$.

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- Repeat for $\mathbf{S}_2, \mathbf{S}_3, \ldots$ to get $F'''''''(x) = \lfloor \mathbf{A}_x \rceil_p = U(x)$. $\square$

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