Session #5: Learning With Errors

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Winter School on Lattice-Based Cryptography and Applications
Bar-Ilan University, Israel
19 Feb 2012 – 22 Feb 2012

Last Time...

- SIS: find "small" nontrivial $z_1, \ldots, z_m \in \mathbb{Z}$ such that:

$$\begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} \quad \begin{pmatrix} | \\ \mathbf{a}_2 \\ | \end{pmatrix} \quad \cdots \quad \begin{pmatrix} | \\ \mathbf{a}_m \\ | \end{pmatrix} \quad \in \mathbb{Z}_q^n$$

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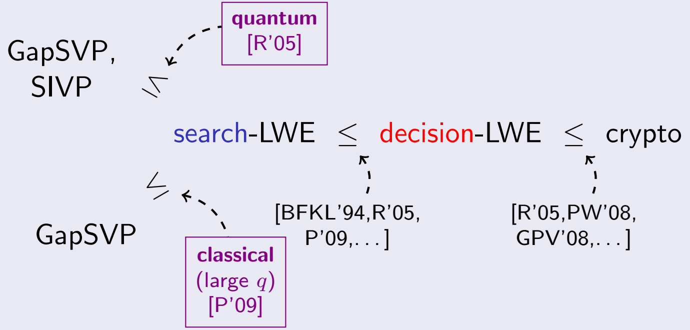
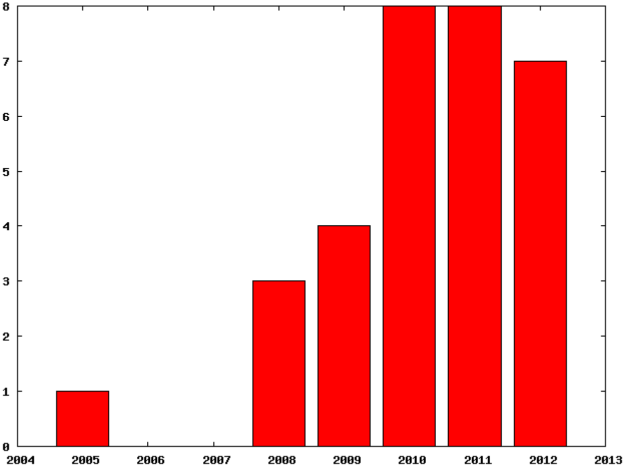
- This talk: a complementary problem, Learning With Errors

Overview of LWE Hardness



History of LWE

Crypto papers with “something new” regarding LWE:



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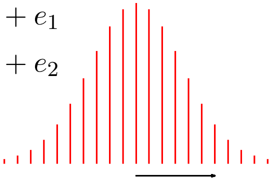

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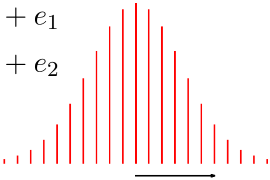

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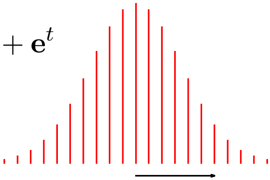

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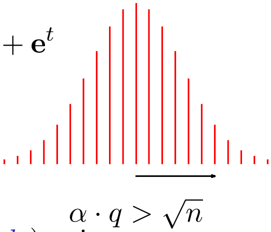

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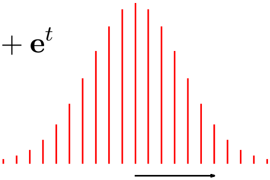

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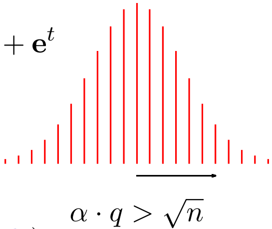

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  - ⋆ There's an $\exp((\alpha q)^2)$-time attack! [AG'11]

SIS versus LWE

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'CRYPTOMANIA'

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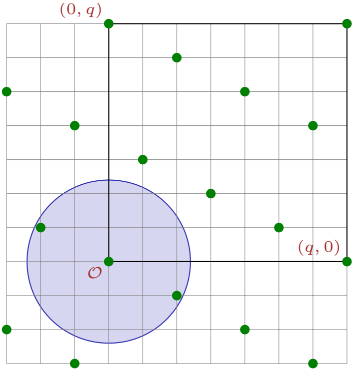
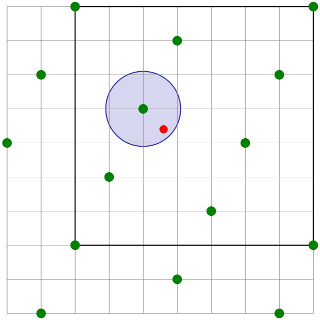
Average-case SVP:	Average-case BDD:
$\mathcal{L}^{\perp}(\mathbf{A}) = \{\mathbf{z} \in \mathbb{Z}^m \ : \ \mathbf{A}\mathbf{z} = \mathbf{0}\}$	$\mathcal{L}(\mathbf{A}) = \{\mathbf{z}^t \equiv \mathbf{s}^t\mathbf{A} \bmod q\}$





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$$\begin{aligned} \mathbf{a}\ ,\ b' &= b + \langle \mathbf{t}, \mathbf{a} \rangle \\ &= \langle \mathbf{s} + \mathbf{t}, \mathbf{a} \rangle + e. \end{aligned}$$

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3. Multiple secrets: $(\mathbf{a}, b_1 \approx \langle \mathbf{s}_1, \mathbf{a} \rangle, \ldots, b_t \approx \langle \mathbf{s}_t, \mathbf{a} \rangle)$ vs. $(\mathbf{a}, b_1, \ldots, b_t)$.

   Simple hybrid argument, since $\mathbf{a}$'s are *public*.

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- Suppose $\mathcal{D}$ solves decision-LWE: it 'perfectly' distinguishes between pairs $(\mathbf{a}, b = \langle \mathbf{s}, \mathbf{a} \rangle + e)$ and $(\mathbf{a}, b)$.

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- Notice: $b = \langle \mathbf{s}, \mathbf{a}' \rangle + s_1 \cdot r + e$.

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- If $\boxed{q = \text{poly}(n)}$, to find $s_1 \in \mathbb{Z}_q$ it suffices to test whether $s_1 \stackrel{?}{=} 0$, because we can shift s_1 by $0, 1, \ldots, q-1$. Same for $s_2, s_3, \ldots, s_n$.

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- Don't really need $\boxed{\text{prime } q = \text{poly}(n)}$ [P'09,ACPS'09,MM'11,MP'12]

Decision-LWE with 'Short' Secrets

Theorem ([M'01,ACPS'09])

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- This maps $(\mathbf{a}, b)$ to $(\mathbf{a}', b')$, so it applies to decision-LWE too.

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$\mathbf{s} \leftarrow \mathbb{Z}_q^n$

$\mathbf{A} \leftarrow \mathbb{Z}_q^{n \times m}$

(Images courtesy xkcd.org)

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$(\mathbf{A}, \mathbf{b}^t), (\mathbf{u}, u')$ by LWE

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$(\mathbf{A}, \mathbf{b}^t), (\mathbf{u}, u')$

by LWE and

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Primal | Dual

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(shared) $\mathbf{A}$ size: $n \times (n \log q)$ elements of $\mathbb{Z}_q$

(user) pk & ct size: $n \log q$ & n elements, or vice-versa

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$\mathbf{s} \leftarrow \chi^n$

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by LWE (HNF)

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$b' - \mathbf{s}^t \, \mathbf{b} \approx \text{bit} \cdot \frac{q}{2}$

$(\mathbf{A}, \mathbf{u}, \mathbf{b}, b')$

by LWE (HNF)

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When We Come Back. . .

- A different kind of LWE application: Efficient pseudorandom functions

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Selected bibliography for this talk:

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