One-Way Functions and Hardcore Predicates

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Bar-Ilan Winter School

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Today's Plan

1. One-way functions and hardcore predicates
2. Pseudorandom generators
3. Pseudorandom functions and permutations
4. Symmetric encryption and MACs.

Online Material

Books:

- Oded Goldreich. Foundations of Cryptography
  http://www.wisdom.weizmann.ac.il/~oded/foc-book.html

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Lecture notes:

- Ran Canetti http://www.cs.tau.ac.il/~canetti/f08.html
- Iftach Haitner http://www.cs.tau.ac.il/~iftachh/Courses/FOC/Spring13/index.html
- Yehuda Lindell http://u.cs.biu.ac.il/~lindell/89-856/main-89-856.html
- Luca Trevisan http://www.cs.berkeley.edu/~daw/cs276/
- Salil Vadhan
  http://people.seas.harvard.edu/~salil/cs120/

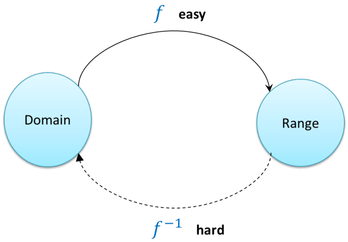
Before we Begin

- We assume basic knowledge of probability theory and computational models, yet please ask us if something is unclear
- We sometimes skip some details (left as exercises for you :-)) and sometimes slightly cheat (we'll clearly mark when)
- Slides are slightly different from your version.
- Please ask questions

Part I

One-Way Functions

Informal Discussion



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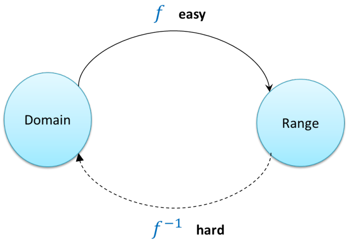

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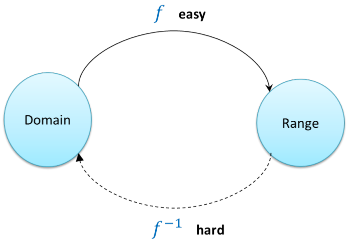

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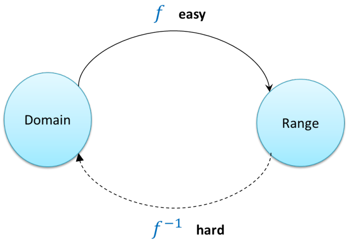

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- Why should we care about OWFs?
- Hidden in (almost) any cryptographic primitive: necessary for "cryptography"
- Sufficient for many cryptographic primitives

Formal Definition

Definition 1 (one-way functions (OWFs))

A polynomial-time computable function $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ is one-way, if

$$\Pr_{x \xleftarrow{R} \{0,1\}^n} \left[A(1^n, f(x)) \in f^{-1}(f(x)) \right] = \text{neg}(n)$$

for any PPT A.

polynomial-time computable: there exists polynomial-time algorithm F, such that $F(x) = f(x)$ for every $x \in \{0,1\}^*$

neg: a function $\mu \colon \mathbb{N} \mapsto [0,1]$ is a negligible function of n, denoted $\mu(n) = \text{neg}(n)$, if for any $p \in \text{poly}$ there exists $n' \in \mathbb{N}$ such that $\mu(n) < 1/p(n)$ for all $n > n'$

$x \xleftarrow{R} \{0,1\}^n$: x is uniformly drawn from $\{0,1\}^n$

PPT: probabilistic polynomial-time algorithm

We typically omit 1^n from the input list of A

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- Efficiently computable function
- Hard on the average
- Negligible inversion probability (i.e., $< 1/\operatorname{poly}(n)$)
- PPT— probabilistic polynomial-time algorithm
- Asymptotic

Non-Uniform OWFs

Definition 2 (non-uniform OWFs)

A polynomial-time computable function $f : \{0,1\}^* \mapsto \{0,1\}^*$ is non-uniformly one-way, if

$$\Pr_{x \xleftarrow{R} \{0,1\}^n} \left[C_n(f(x)) \in f^{-1}(f(x)) \right] = \operatorname{neg}(n)$$

for any polynomial-size family of circuits $\{C_n\}_{n \in \mathbb{N}}$.

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We will mainly focus on uniform security

Length Preserving OWF

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A function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is length preserving, if $|f(x)| = |x|$ for every $x \in \{0,1\}^*$

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Convention for rest of the talk

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a one-way function

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A poly-time computable function $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ is α-one-way, if

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- (strong) OWF according to Definition 1, is neg-one-way according to the above definition
- Can we convert (i.e., amplify) weak OWFs into strong ones?

Strong to Weak OWFs

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Assume there exists OWFs, then there exist functions that are $\frac{2}{3}$-one-way, but not (strong) one-way

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Proof:

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Proof: For a OWF f, let

$$g(x) = \begin{cases} (1, f(x)), & x_1 = 1; \\ 0, & \text{otherwise } (x_1 = 1). \end{cases}$$

Weak to Strong OWFs

Theorem 7 (weak to strong OWFs (Yao))

Assume there exist $(1 - \delta)$*-weak OWFs with* $\delta(n) \geq 1/q(n)$ *for some* $q \in \mathrm{poly}$*, then there exist (strong) one-way functions.*

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Assume there exist $(1-\delta)$-weak OWFs with $\delta(n) \geq 1/q(n)$ for some $q \in \text{poly}$, then there exist (strong) one-way functions.

- Idea: parallel repetition (i.e., direct product): Consider $g(x_1, \ldots, x_t) = f(x_1), \ldots, f(x_t)$ for large enough t

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  Consider matrix multiplication: Let $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$

  Computing Ax takes $\Theta(n^2)$ times, but computing $A(x_1, x_2, \ldots, x_n)$ takes ...

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- Fortunately, parallel repetition does amplify weak OWFs :-)

Amplification via Parallel Repetition

Theorem 8

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$*, and for* $t(n) := \left\lceil \frac{\log^2 n}{\delta(n)} \right\rceil$ *define* $g\colon (\{0,1\}^n)^{t(n)} \mapsto (\{0,1\}^n)^{t(n)}$ *as*

$$g(x_1,\ldots,x_{t(n)}) = f(x_1),\ldots,f(x_{t(n)})$$

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In the following we fix (an assumed) PPT A, $p \in \mathrm{poly}$ and infinite set $\mathcal{I} \subseteq \mathbb{N}$ s.t.

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for every $n \in \mathcal{I}$. We also “fix" $n \in \mathcal{I}$ and omit it from the notation.

Proving that g is One-Way – the Naive Approach

Assume A attacks each of the t outputs of g independently: $\exists$ PPT A′ such that $A(z_1, \ldots, z_t) = A'(z_1) \ldots, A'(z_t)$

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It follows that A′ inverts f with probability greater than $(1 - \delta(n))$.

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Otherwise

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A less naive approach would be to assume that A goes over the inputs sequentially.

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Assume A attacks each of the t outputs of g independently: $\exists$ PPT A$'$ such that $\mathsf{A}(z_1,\ldots,z_t) = \mathsf{A}'(z_1)\ldots,\mathsf{A}'(z_t)$

It follows that A$'$ inverts f with probability greater than $(1-\delta(n))$. Otherwise

$$\Pr_{w \xleftarrow{R} \{0,1\}^{t(n)\cdot n}}[\mathsf{A}(g(w)) \in g^{-1}(g(w))] = \prod_{i=1}^{t} \Pr_{x \xleftarrow{R} \{0,1\}^n}\left[\mathsf{A}'(f(x)) \in f^{-1}(f(x))\right]$$
$$\leq (1-\delta(n))^{t(n)} \leq e^{-\log^2 n} \leq n^{-\log n}$$

Hence A$'$ violates the weak hardness of f

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Unfortunately, we can assume none of the above.

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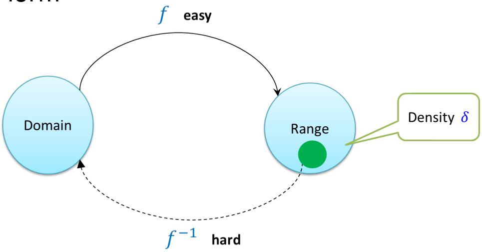
Any idea?

Hardcore Sets

Assume f is of the form



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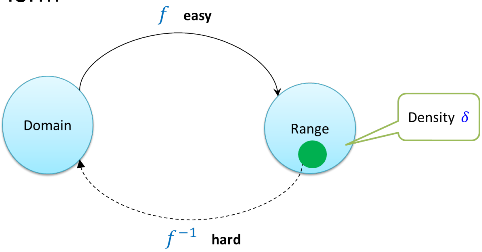

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$\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$ is a δ-hardcore set for $f \colon \{0,1\}^n \mapsto \{0,1\}^n$, if:

1. $\Pr_{x \xleftarrow{R} \{0,1\}^n}[f(x) \in \mathcal{S}] \geq \delta(n)$ for large enough n, and
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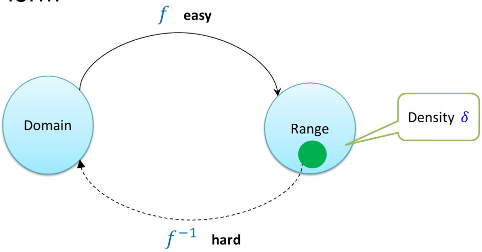

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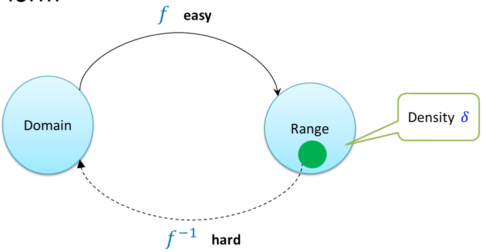

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Unfortunately, we do not know how to prove that f has hardcore set :-<

Failing Sets

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A function $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ has a δ-failing set for a pair (A, q) of algorithm and polynomial, if exists $\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$, such that the following holds for large enough n:

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Let f be a $(1-\delta)$-OWF, then f has a $\delta/2$-failing set, for any pair of PPT A and $q \in \text{poly}$.

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We'll use A to contradict the hardness of f.

Using A to Invert f

For $n \in \mathbb{N}$, let $\mathcal{S}_n := \{y \in \{0,1\}^n \colon \Pr\left[\mathsf{A}(y) \in f^{-1}(y)\right] < 1/q(n)\}$.

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Do (with fresh randomness) for $n \cdot q(n)$ times:
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Hence, for large enough $n \in \mathcal{I}$: $\Pr_{x \xleftarrow{R} \{0,1\}^n}\left[\mathsf{B}(f(x)) \in f^{-1}(f(x))\right] > 1 - \delta(n)$.

Namely, f is not $(1-\delta)$-one-way $\square$

Proving g is One-Way cont.

We show that is g is not one way, then f has no $\delta/2$ flailing-set for some PPT B and $q \in$ poly.

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Assume $\exists$ PPT A, $p \in$ poly and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that

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Namely, f has no $\delta/2$ failing set for $(\mathsf{B}, q = 2t(n)p(n))$

The No Failing-Set Algorithm

Algorithm 16 (Inverter B on input $y \in \{0,1\}^n$)

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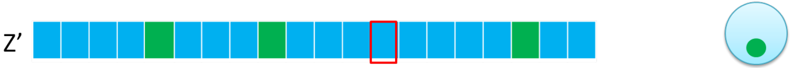

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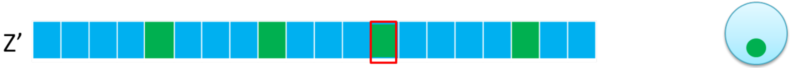

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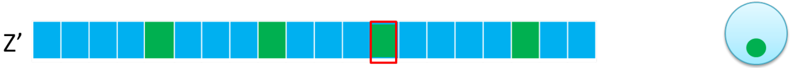

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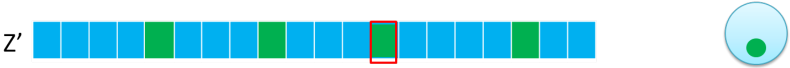

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To conclude the proof take $\mathcal{L} = \{v \in \{0,1\}^{t(n) \cdot n} \colon \mathsf{A}(v) \in g^{-1}(v)\}$

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- What properties of the weak OWFs have we used in the proof?

Part II

Hardcore Predicates

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Such functions have many cryptographic applications

Formal Definition

Definition 18 (hardcore predicates)

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  Answer to above is positive, in case f is one-to-one

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Assuming f *is one way, then*

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The resulting predicate is not for f but for (the one-way function) g^t ...

The Goldreich-Levin Hardcore predicate

For $x, r \in \{0,1\}^n$, let $\langle x, r \rangle_2 := (\sum_{i=1}^{n} x_i \cdot r_i) \bmod 2 = \bigoplus_{i=1}^{n} x_i \cdot r_i$.

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- Note that if f is one-to-one, then so is g.

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For $x, r \in \{0,1\}^n$, let $\langle x, r\rangle_2 := (\sum_{i=1}^{n} x_i \cdot r_i) \bmod 2 = \bigoplus_{i=1}^{n} x_i \cdot r_i$.

Theorem 20 (Goldreich-Levin)

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, define $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ as $g(x, r) = (f(x), r)$.

If f is one-way, then $b(x, r) := \langle x, r\rangle_2$ is an hardcore predicate of g.

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Proof by reduction: a PPT A for predicting $b(x,r)$ “too well" from $(f(x), r)$, implies an inverter for f

Proving Goldreich-Levin Theorem

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Assume $\exists$ PPT A, $p \in$ poly and infinite set $\mathcal{I} \subseteq \mathbb{N}$ with

$$\Pr[\mathsf{A}(g(U_n, R_n)) = b(U_n, R_n)] \geq \frac{1}{2} + \frac{1}{p(n)},$$

for any $n \in \mathcal{I}$, where U_n and R_n are uniformly (and independently) distributed over $\{0, 1\}^n$.

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Claim 21

For $n \in \mathcal{I}$, there exists a set $\mathcal{S}_n \subseteq \{0,1\}^n$ with

1. $\frac{|\mathcal{S}_n|}{2^n} \geq \frac{1}{2p(n)}$, and
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We next show $\exists$ PPT B and $q \in$ poly with

$$\Pr\left[B(f(x)) \in f^{-1}(f(x))\right] \geq \frac{1}{q(n)},$$

for every $n \in \mathcal{I}$ and $x \in \mathcal{S}_n$.

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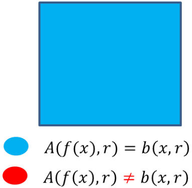
In the following we fix $n \in \mathcal{I}$ and $x \in \mathcal{S}_n$.

The Perfect Case

$$\Pr\left[\mathsf{A}(f(x), R_n) = b(x, R_n)\right] = 1$$



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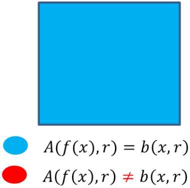
$$\Pr\left[A(f(x), R_n) = b(x, R_n)\right] = 1$$



In particular, $A(f(x), e^i) = b(x, e^i)$ for every $i \in [n]$, where $e^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})$.

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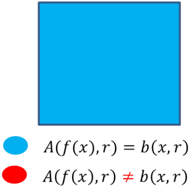

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Hence, $x_i = \langle x, e^i \rangle_2$

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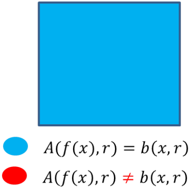

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Hence, $x_i = \langle x, e^i \rangle_2 = b(x, e^i) = \mathsf{A}(f(x), e^i)$

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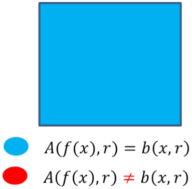
$\Pr[A(f(x), R_n) = b(x, R_n)] = 1$



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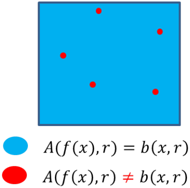
Hence, $x_i = \langle x, e^i \rangle_2 = b(x, e^i) = A(f(x), e^i)$

Let $B(y) = (A(y, e^1), \ldots, A(y, e^n))$

Easy case

$$\Pr\left[A(f(x), R_n) = b(x, R_n)\right] \geq 1 - \text{neg}(n)$$



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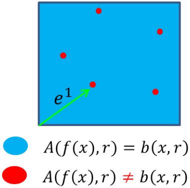

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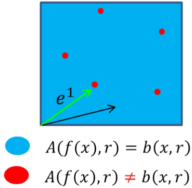

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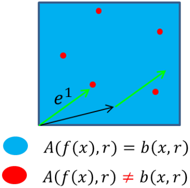

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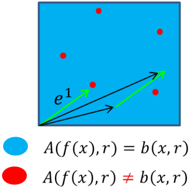

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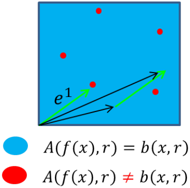
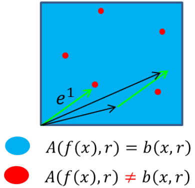


1. $b(x, w) \oplus b(x, y) = b(x, w \oplus y)$ for every $w, y \in \{0, 1\}^n$.

Easy case

$$\Pr[A(f(x), R_n) = b(x, R_n)] \geq 1 - \mathrm{neg}(n)$$



1. $b(x, w) \oplus b(x, y) = b(x, w \oplus y)$ for every $w, y \in \{0,1\}^n$.
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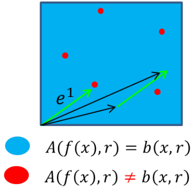

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Hence, $\forall i \in [n]$:

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Easy case

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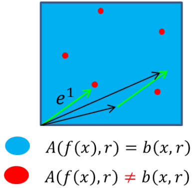

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1. $x_i = b(x, e^i) = b(x, r) \oplus b(x, r \oplus e^i)$ for every $r \in \{0, 1\}^n$
2. $\Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \geq 1 - \text{neg}(n)$

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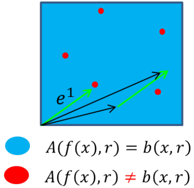

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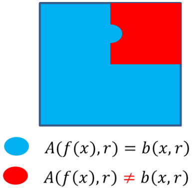
Algorithm 22 (Inverter B on input y)

Return $(\mathsf{A}(y, R_n) \oplus \mathsf{A}(y, R_n \oplus e^1)), \ldots, \mathsf{A}(y, R_n) \oplus \mathsf{A}(y, R_n \oplus e^n))$.

Intermediate Case

$$\Pr[A(f(x), R_n) = b(x, R_n)] \geq \frac{3}{4} + \frac{1}{q(n)}$$



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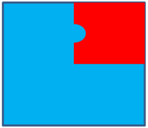

For any $i \in [n]$

$$\begin{aligned} &\Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i] \\ &\quad \geq \quad \Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \end{aligned}$$

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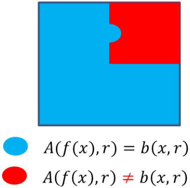

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$$\begin{aligned}
&\Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i] \\
&\quad \geq \Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \\
&\quad \geq 1 - \left(1 - (\frac{3}{4} + \frac{1}{q(n)})\right) - \left(1 - (\frac{3}{4} + \frac{1}{q(n)})\right)
\end{aligned}$$

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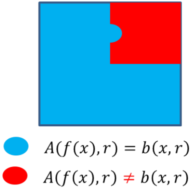
$$\Pr\left[\mathrm{A}(f(x), R_n) = b(x, R_n)\right] \geq \tfrac{3}{4} + \tfrac{1}{q(n)}$$



For any $i \in [n]$

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&\quad \geq \quad 1 - \left(1 - (\frac{3}{4} + \frac{1}{q(n)})\right) - \left(1 - (\frac{3}{4} + \frac{1}{q(n)})\right) = \frac{1}{2} + \frac{2}{q(n)}
\end{aligned}$$

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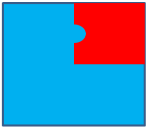

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\end{aligned}$$

Algorithm 23 (Inverter B on input $y \in \{0,1\}^n$)

1. For every $i \in [n]$
   1. Sample $r^1, \ldots, r^v \in \{0,1\}^n$ uniformly at random
   2. Let $m_i = \text{maj}_{j \in [v]}\{(A(y, r^j) \oplus A(y, r^j \oplus e^i)\}$
2. Output $(m_1, \ldots, m_n)$

B's Success Provability

The following holds for "large enough" $v = v(n) \in \text{poly}(n)$.

Claim 24

For every $i \in [n]$, it holds that $\Pr[m_i = x_i] \geq 1 - \text{neg}(n)$.

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Proof: For $j \in [v]$, let the indicator rv W^j be 1, iff $A(f(x), r^j) \oplus A(f(x), r^j \oplus e^i) = x_i$.

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We want to lowerbound $\Pr\left[\sum_{j=1}^{v} W^j > \frac{v}{2}\right]$.

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We want to lowerbound $\Pr\left[\sum_{j=1}^{v} W^j > \frac{v}{2}\right]$.

- The W^j are iids and $\text{E}[W^j] \geq \frac{1}{2} + \frac{2}{q(n)}$ for every $j \in [v]$

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Proof: For $j \in [v]$, let the indicator rv W^j be 1, iff $A(f(x), r^j) \oplus A(f(x), r^j \oplus e^i) = x_i$.
We want to lowerbound $\Pr\left[\sum_{j=1}^{v} W^j > \frac{v}{2}\right]$.

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Lemma 25 (Hoeffding's inequality)

Let $X^1, \ldots, X^v$ be iids over $[0,1]$ with expectation μ. Then,
$\Pr\left[\left|\frac{\sum_{j=i}^{v} X^j}{v} - \mu\right| \geq \varepsilon\right] \leq 2 \cdot \exp(-2\varepsilon^2 v)$ *for every* $\varepsilon > 0$.

B's Success Provability

The following holds for "large enough" $v = v(n) \in \text{poly}(n)$.

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We complete the proof taking $X^j = W^j$, $\varepsilon = 1/4q(n)$ and $v \in \omega(\log(n) \cdot q(n)^2)$.

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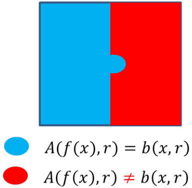

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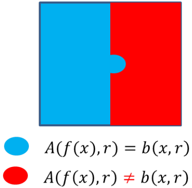

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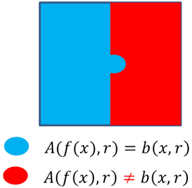

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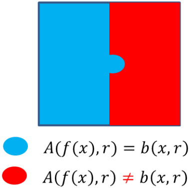

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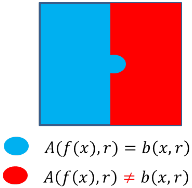

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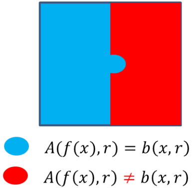

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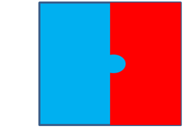

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